Financial Intermediation Cycles without Fire Sales*

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Abstract

Under financing frictions, negative shocks have a lasting impact on credit intermediaries' net worth and lending capacity. Anticipating tighter credit-supply conditions and the resulting difficulty in financing ongoing capital growth, firms' current incentives to borrow and create productive capital weaken. Such credit-demand contraction reduces intermediaries' profitability, delays their rebuilding of net worth, and traps the economy in downturns. This paper develops a model of credit cycles featuring seemingly stable booms, persistent crises, and transitions driven by intermediaries' state-dependent leverage choices. Unlike in the fire-sale frameworks (e.g., Kiyotaki and Moore, 1997), credit is not tied to secondary-market prices of collateral assets.

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1 Introduction

Financial crises are marked by net worth destruction, the tightening of borrowing constraints, and ensuing credit crunches (e.g., Bernanke, Gertler, and Gilchrist (1999)). Following Kiyotaki and Moore (1997) (hereafter KM), models of credit cycles and financial crises often rely on asset fire sale as a shock amplification mechanism (Shleifer and Vishny, 2011). Before the Global Financial Crisis (GFC), the focus was on borrowers' net-worth losses. After the GFC, fire-sale mechanisms have applied to amplify the impact of shocks on net worth of intermediaries that supply credit.

Kocherlakota (2000) pointed out that in KM's deterministic environment, the one-time shock cannot cause a severe asset fire sale and large crisis, as the economy is known to recover.[1] A new generation of models address this issue by analyzing stochastic environments with recurrent shocks that allows risk premia to play a critical role: after negative shocks, risk premia rise, causing asset prices to slump (e.g., He and Krishnamurthy (2013), Brunnermeier and Sannikov (2014)).

A conflict emerges between crisis severity and duration in these models of intermediation cycles: higher risk premia in crises raise expected asset returns, accelerating net worth recovery—that is, investment opportunities improve in crises (see also Gersbach, Rochet, and Scheffel (2023)). Modeling credit crunches via fire sale also faces an empirical challenge. Credit is tied to the market prices of collateral assets. The mechanism's empirical relevance depends on the prevalence of debt backed by marketable assets, which constitutes about 20% of firms' debts (Lian and Ma, 2020).[2]

This paper develops a model of credit intermediation cycles without fire sales. Crisis severity and duration reinforce one another, as investment opportunities deteriorate rather than improve for financial intermediaries in crises. Recovery is fragile and prone to double-dip recessions. Beyond characterizing crises, the fully stochastic environment allows for a complete analysis of boom-bust cycles and endogenous transition across cycle stages. These equilibrium features are analytically characterized—unlike much of the macro-finance literature that depends on numerical solutions. In the following, the model is presented in three steps: first, crisis dynamics are characterized; next, transition from crisis to boom is examined; and finally, properties of the boom are discussed.

[1]Given that asset prices rise back for sure, it takes a small drop in asset prices to make the assets attractive to hold.
[2]Related, Benmelech, Kumar, and Rajan (2024) find a secular decline in the U.S. firms' reliance on secured debt.

Firms rely on credit supplied by financial intermediaries to fund the creation and growth of productive capital. In line with evidence (e.g., Baron, Verner, and Xiong (2021)), intermediaries' equity plays a pivotal role, acting as the state variable. When negative shocks erode their equity, intermediaries' balance-sheet capacity shrinks because they cannot recapitalize frictionlessly. As a result, credit supply contracts. What happens next to credit demand is critical: anticipating the shocks' impact to be persistent, firms' demand for credit declines. The reason is that because firms expect future credit availability to worsen and capital to grow slowly as a result, firms' current incentive to build capital weakens. Thus, intermediaries' investment opportunities (i.e., firms' credit demand) deteriorate in crises. This is opposite to models of financial intermediation where intermediaries' investment opportunities improve when intermediaries become undercapitalized.

The forward-looking credit demand is realistic. In practice, when deciding whether to initiate projects, firms consider future growth opportunities, the associated financing needs, and credit availability. If they anticipate abundant future credit and faster capital growth, firms want to build capital now and increase their current demand for credit. Conversely, expectations of future credit contraction and slower growth weaken their current incentive to build capital and credit demand.

The other key model ingredient is that intermediaries face equity issuance costs and optimally raise equity only when they are severely undercapitalized. Except when the economy is at the lower (optimal issuance) boundary of intermediaries' equity, intermediaries do not recapitalize immediately following negative shocks, which is the reason why the shocks' impact persists.

Crises originate on the credit-supply side: because intermediaries cannot be frictionlessly recapitalized, negative shocks reduce their current equity and shift downward its future trajectory through the persistent impact. Credit demand contracts in response, reducing intermediaries' profitability and slowing their equity rebuild.[3] Firms' concern over future credit availability for capital growth links the trajectory of intermediaries' equity and their current lending opportunities.[4] The

[3]Even though the equilibrium loan rate rises in crises as undercapitalized intermediaries demand a higher risk premium, credit-demand contraction (i.e., an inward shift of demand curve) leads to a significant decline in the equilibrium quantity of credit, hurting intermediaries' lending profits that depend on both the price and quantity of credit.

[4]The persistence of shock impact under financial constraints has been well established, first for nonfinancial firms (Bernanke, Gertler, and Gilchrist, 1999) and later for banks and other financial intermediaries (Brunnermeier and Sannikov, 2016a). The innovation in my model is to link the persistent impact of shocks on intermediary equity (the credit-supply side) to the endogenous variation of credit demand (i.e., the intermediaries' investment opportunities).

severity and duration of crises are mutually reinforcing.[5] On the one hand (from crisis severity to duration), a crisis is more persistent when credit-demand contraction is more severe, implying sluggish rebuild of intermediaries' equity. On the other hand (from crisis duration to severity), when anticipating a persistently low level of intermediary equity, credit tightening, and the slowdown of capital growth, firms want to create less capital now, so their credit needs weaken, resulting in a low equilibrium level of credit and growth. In the model, credit-demand contraction during downturns is driven not by sentiment but by rational expectations of persistent credit-supply tightening.

Intermediaries' equity or "net worth" is the state variable. Intermediaries' choice of leverage determines its shock sensitivity. Therefore, characterizing intermediary leverage is essential for analyzing the equilibrium dynamics. The model has a mechanism through which intermediaries' leverage choice prolongs crises. Following negative shocks, the deterioration of lending opportunities triggers intermediaries' deleveraging. Reducing shock exposure seems beneficial, but the low leverage primarily dampens positive shocks rather than negative ones. When intermediaries' equity is already low in crises, more negative shocks eventually trigger equity issuance. As a reminder, intermediaries face issuance costs but optimally raise equity when they are severely undercapitalized (when the marginal value of equity is high). Issuance limits the impact of negative shocks as equity cannot fall below the issuance threshold. Thus, the shock dampening effect of low leverage in crises mainly applies to positive shocks that could help rebuild intermediary equity.

Intermediaries' leverage choice renders recovery from crises not only sluggish but fragile. As intermediaries rebuild equity, firms foresee credit-supply expansion and faster capital growth, which boosts their incentives to create capital now and stimulates credit demand. Improved lending opportunities induce intermediaries to lever up. Since their' equity has risen and is no longer close to the level that triggers issuance, the impact of negative shocks is not as limited by prospective equity raising as it is at the depth of crises. Thus, as leverage builds up along the recovery path, negative shocks can quickly erode intermediaries' equity, resulting in a double-dip recession.

Next, consider the transition into a boom. In the absence of consecutive negative shocks, intermediaries rebuild equity along the recovery path, and the equilibrium level of credit provision

[5]Shock amplification and the persistence of shocks' impact are two key themes in the literature on economic fluctuations. Brunnermeier, Eisenbach, and Sannikov (2013) provide a comprehensive survey.

increases, allowing firms to create and grow capital at a faster rate. Intermediaries' leverage also rises as previously noted, and it will continue rising until firms reach their borrowing limit—that is, until the level of intermediaries' equity and associated credit provision are so high that firms have pledged all future cash flows to borrow from intermediaries. Beyond this point, further increases in intermediary equity can no longer trigger rapid expansion of firms' credit demand, and intermediary leverage turns countercyclical: any further increase in intermediaries' equity now reduces leverage, because as the asset side of intermediaries' balance sheets (lending) no longer expands as fast, additional equity on the liability side replaces debt, lowering leverage.

In summary, intermediary leverage exhibits hump-shaped dynamics. When intermediary equity—the state variable—is low, leverage rises as equity increases, showing procyclicality. The driving force is the positive feedback from a higher intermediary equity (and anticipated future credit-supply expansion) to a stronger credit demand. Capital is both productive and carries growth options, so the demand for credit that finances capital creation is driven by expectations of future credit-supply conditions that affect capital growth. This is about how much firms want to borrow. When intermediary equity is sufficiently high, what constrains the equilibrium level of credit provision is no longer the inelastic supply but firms' borrowing capacity (pledgeable value). This is about how much firms can borrow. In this region, intermediary leverage is countercyclical.

Intermediary leverage dynamics are important for understanding the credit cycle. The hump-shaped pattern implies that the economy is insensitive to shocks (intermediary leverage is low) both when intermediary equity is low (crises) and when it is high (booms). The economy tends to be trapped in the two extremes—that is, the stationary distribution of the state variable is bimodal. In the region with intermediate levels of intermediary equity, the system spends relatively less time as it's shock-sensitive (intermediary leverage is high). Thus, despite evolving along continuous paths (shocks are Brownian), the economy displays two endogenous regimes. These results speak to the findings on the multimodality of financial conditions (Adrian, Boyarchenko, and Giannone, 2021).

It is worth noting that intermediaries in the model represent both banks and non-bank lenders that employ leverage and face frictions when raising equity capital. In practice, the key differences between banks and non-bank lenders often stem from their regulatory environments and from their investment and funding clienteles. In the model's laissez-faire setting, however, these intermediaries do not face regulations. Moreover, the characteristics of credit demand in the model are not unique to any particular sector of the economy or to borrowers with specific risk profiles.

Finally, the model offers new perspectives on credit policies. It has become increasingly common for governments to lend directly to firms in crises, surpassing financial intermediaries.[6] However, such policy cannot significantly increase the equilibrium level of credit provision, if credit demand is weak. Unless the government replaces intermediaries entirely, the marginal supply of credit still come from intermediaries. Consequently, firms' expectations of capital growth and current incentives to borrow and invest remain tied to the trajectory of intermediary equity. The model shows that policy intervention prolongs crises by further eroding intermediaries' lending opportunities. If part of credit demand is met by the government, intermediaries earn profits only on the residual demand, so their equity rebuild is slower than the laissez-faire case.

Government intervention renders the recovery sluggish also through its impact on intermediary leverage. As noted earlier, deleveraging slows down recovery. By reducing intermediaries' lending opportunities, policy intervention strengthens intermediaries' incentive to deleverage.

Government intervention also makes recovery more fragile by amplifying the procyclicality of intermediary leverage in the low-equity region. Intermediaries maintain sufficient profits over credit cycles to cover the occasionally incurred equity issuance costs. To do so, they raise leverage in good times—when government intervention is absent and lending opportunities are more abundant—and reduce leverage in bad times when intervention erodes lending profits. This response to intervention leads to more pronounced leverage procyclicality. Consequently, along the recovery path, intermediaries' leverage rises more sharply. When magnified by a higher leverage, negative shocks can erode intermediaries' equity quickly and push the economy back into crises.

Literature. This paper contributes to the literature on financial accelerators (e.g., Bernanke and Gertler (1989); Carlstrom and Fuerst (1997); Kiyotaki and Moore (1997); Bernanke, Gertler, and Gilchrist (1999); Jermann and Quadrini (2012)). After the GFC, attention has shifted from produc-

[6]Examples include the Primary and Secondary Market Corporate Credit Facilities that were introduced in the U.S. during the Covid-19 pandemic, and corporate bond purchases at the Bank of Japan and European Central Bank.

ers' net-worth destruction to the vulnerabilities of financial intermediaries. In models of financial crises, fire sales and volatile asset prices serve as the primary shock-amplification mechanisms.[7] This modeling approach lays the foundation for a broad literature on financial regulations and crisis-intervention policies.[8] In the opening section of this article, I explain the theoretical and empirical challenges confronting the fire-sale approach, particularly in its application to credit cycles.

This paper provides a model of financial intermediation cycles without fire sales. Rampini and Viswanathan (2019) (hereafter RV) represent another approach that also departs from the fire-sale framework.[9] They model borrowers' and intermediaries' net worth jointly as state variables.[10] In my model, intermediaries' net worth is the only state variable. Firms face a natural debt limit—they cannot pledge more than all future cash flows—but, unlike RV, they are not subject to haircuts or margin requirements that make their net worth a second state variable. My focus is on analyzing the dynamics of intermediaries' net worth when firms' demand for credit varies with the anticipated credit-supply conditions. Moreover, my model is fully stochastic, generating booms, crises, and endogenous transitions, and it characterizes intermediaries' leverage choice that depends on ex ante risk-return assessment, varies across credit-cycle stages, and affects the cross-stage transitions.[11]

Credit demand in the model has two features. The first is about how much firms *want to borrow*. Capital carries growth opportunities, so the demand for credit that finances capital creation is driven by expectations of future credit-supply conditions that affect growth. When intermediaries' net worth is low, this feature creates a comovement between intermediaries' net worth and lending opportunities. It locks the economy in crises with both depressed credit supply and demand. It also contributes to the procyclicality of intermediaries' leverage along the recovery path, making

[7] Examples include Lorenzoni (2008); Adrian and Boyarchenko (2012); He and Krishnamurthy (2012, 2013); Brunnermeier and Sannikov (2014, 2016b); Dávila and Korinek (2017); Di Tella (2017, 2019); Moreira and Savov (2017); and Caballero and Simsek (2020). Please refer to Brunnermeier and Sannikov (2016a) for a review.

[8] Examples—by no means exhaustive—include Gertler and Kiyotaki (2010); Gertler and Karadi (2011); Phelan (2016); Korinek and Simsek (2016); Caballero and Simsek (2020, 2021); and Akinci and Queralto (2022).

[9] RV's collateral constraint can be broadly viewed as a pledgeability constraint. Its role is not to connect credit and asset fire sale. Their model generates crises without relying on a slump of market prices of collateral assets.

[10] Holmström and Tirole (1997) model jointly borrowers' and intermediaries' net worth without offering a dynamic framework. Beyond RV, recent dynamic models emphasize quantitative performances (e.g., Gete (2018); Villacorta (2018); Ferrante (2019); Elenev, Landvoigt, and Nieuwerburgh (2021); and Mendicino et al. (2025)).

[11] RV analyze dynamics around one-time shocks as in other papers (e.g., KM) that derive analytical (rather than numerical) results. My paper also delivers analytical results on credit cycles but does so in a fully stochastic setting.

recovery fragile. The second feature is about how much firms *can borrow*. The limit is present value of future cash flows. When intermediaries' net worth is high and credit is abundant, firms hit borrowing limits that constrain intermediaries' lending (asset expansion). As a result, when intermediaries accumulate net worth, equity replaces debt, making leverage countercyclical. The hump-shaped leverage dynamics imply that between stable booms and stagnant crises, transition can be abrupt, because intermediaries' leverage and the shock sensitivity of their net worth are highest when intermediaries' net worth is at the intermediate levels. Sharp transitions from credit booms to crises and stagnant recovery have been documented in the empirical literature (e.g., Schularick and Taylor (2012); Baron, Verner, and Xiong (2021); and Krishnamurthy and Muir (2025)).

The borrowers' debt limit plays a role that is distinct from that in other models of financial crises and cycles. The literature emphasizes binding debt limits in bad times (e.g., RV).[12] In contrast, firms' borrowing limit in my model (how much firms *can borrow*) binds in good times when credit is abundant. Its impact is on intermediaries' leverage choice that in turn determines the duration of each cycle phase and transitions across phases. In bad times, the relevant force is how much firms *want to borrow*, which drives intermediaries' investment opportunities and profitability.

Models on intermediation cycles generate monotone leverage dynamics, either procyclical or countercyclical.[13] In my model, the cyclicality of intermediary leverage is state-contingent: it is procyclical in certain states and countercyclical in others. The source of leverage cyclicality is also distinct, tied to the aforementioned credit-demand features. In particular, my model shows that the cyclicality of intermediaries' leverage depends on whether their borrower clientele has slack debt capacity. This is relevant for empirical analysis of intermediaries' leverage choice (e.g., Adrian and Shin (2010); He, Kelly, and Manela (2017)). When credit markets are segmented and cycles are not perfectly synchronized across the segments, intermediaries may differ in leverage cyclicality.

[12] Firms' borrowing constraint may take different forms (e.g., Christiano, Motto, and Rostagno (2014); Buera and Moll (2015); Greenwald (2019); Drechsel (2023); Greenwald, Krainer, and Paul (2025)), but the emphasis has been on its role in economic downturns. See Ottonello, Perez, and Varraso (2022) for analysis of generic borrowing constraints.

[13] Countercyclicality arises from improved investment opportunities in crises (e.g., He and Krishnamurthy (2013); Brunnermeier and Sannikov (2014); Phelan (2016); Gersbach, Rochet, and Scheffel (2023)). Risk-based debt limits lead to procyclicality (e.g., Adrian and Boyarchenko (2012); Adrian and Shin (2014); Nuño and Thomas (2017)).

2 Model

Preferences, technology, and endowments are defined first, and then financial frictions are introduced. The continuous-time, infinite-horizon economy has three types of agents, entrepreneurs, households, and bankers. Let $\mathbb{E}$, $\mathbb{H}$, and $\mathbb{B}$ denote the sets of a unit mass of representative entrepreneurs, households, and bankers, respectively.

Entrepreneurs and households consume the numeraire goods and are risk-neutral with discount rate ρ. They maximize life-time consumption. For example, an entrepreneur maximizes

$$\mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t} dc_t^{\mathbb{E}}\right], \tag{1}$$

where $c_t^{\mathbb{E}}$ is the cumulative consumption. Wherever necessary, superscripts denote agents' type.

At $t = 0$, a representative entrepreneur is endowed with k_0 units of productive capital. K_0, the aggregate amount of capital, is $\int_{n\in\mathbb{E}} k_0(n)\, dn$. At time t, the entrepreneur with k_t units of capital can produce $\alpha k_t dt$ units of numeraire goods over dt, where the parameter, α (> 0), represents productivity, so aggregate output is $\alpha K_t dt$. A fraction $\delta dt - \sigma dZ_t$ of capital is destroyed over dt, where Z_t is a Brownian motion. The aggregate shock, dZ_t, is the only source of risk.[14]

In the model, "capital" represents efficiency units that produce goods. It represents projects, business subsidiaries, and productive assets. Let q_t^K denote the present value of all goods generated by one unit of capital. This value incorporates not only the "cash flows", i.e., α per unit of time, but also the options to grow. The entrepreneur has a technology to grow capital. Let i_t denote investment per unit of capital. Investment success is a Poisson event with intensity λ, independent across capital units—that is, a fraction λdt of capital grows at the rate of $F(i_t)$ over dt. $F(\cdot)$ is an increasing and concave function.[15] The net profit is $q_t^K F(i_t) - i_t$, i.e., the value of newly created capital, $q_t^K F(i_t)$, minus the cost, i_t. A fraction $1 - \lambda dt$ of capital does not grow—that is, the investment project turns out to be a zero-profit storage technology (instead of zero profit, a negative return could be introduced but complicates notations without meaningful impact on the

[14]The information filtration is generated by Z_t and satisfies the standard regularity conditions (Protter, 1990).

[15]For a unit of capital, the joint probability of destruction and growth is smaller than dt and thus negligible.

mechanism). Investment is subject to a capacity constraint, $i_t \le \bar{l}$. The following restriction on $F(\cdot)$ is imposed so that capital growth does not exceed the discount rate, ρ, and the equilibrium value of capital, q_t^K, is finite:

$$\rho > \lambda F\left(\bar{l}\right) - \delta + \frac{\sigma^2}{2}. \tag{2}$$

Resources that entrepreneurs need for investment can be obtained from deep-pocket households. The allocation problem in this economy is whether funds can be channeled from households to entrepreneurs. It is assumed that households cannot lend directly to entrepreneurs, and bankers are required to intermediate credit supply. There are many reasons why intermediation is necessary.[16] This paper does not seek to provide another microfoundation. The goal to analyze the dynamics of an economy where credit intermediation is essential. Moreover, "banks" in this paper broadly represent credit intermediaries, including both traditional commercial banks and non-bank lenders. The model applies as long as a credit intermediary engages with risky lending (rather than risk-free investments) and its liability structure contains a mix of debt and equity.

As previously discussed, there are a unit mass of representative bankers. The set of bankers is denoted by $\mathbb{B}$. At $t = 0$, a representative banker is endowed with e_0 units of numeraire goods. E_0, the aggregate amount of bankers' wealth or bank equity, is $\int_{n \in \mathbb{B}} e_0(n)\, dn$. Bankers may issue debt and equity to households, intermediating between households and entrepreneurs ("firms").

After specifying the physical environment (i.e., preferences, technology, and endowments), I now introduce the financial aspects of the economy. Banks extend loans to entrepreneurs at t that are repaid after the entrepreneurs finish their investment projects at $t + dt$. The repayment is $1 + R_t dt$ per unit of goods lent (i.e., the loan rate is R_t), where R_t is to be determined in equilibrium.

Loans are backed by designated capital. Entrepreneurs default on a loan when the backing capital perishes. Let l_t denote the amount of credit for one unit of capital. We have

$$l_t \le q_t^K. \tag{3}$$

As previously discussed, q_t^K is the present value of all cash flows generated by this one unit of

[16] Financial intermediaries have expertise in monitoring (Diamond, 1984), screening (Ramakrishnan and Thakor, 1984), restructuring (Bolton and Freixas, 2000), and enforcing collateralized claims (Rampini and Viswanathan, 2019).

capital and new capital that grows out of it through investments. This is a standard pledgeability constraint.[17] It does not tie credit, l_t, to an asset that is transferred between agents as in a fire-sale mechanism. In fact, throughout credit cycles, entrepreneurs' capital need not exchange hands at all. As noted above, the interpretation of capital is broad, including projects or business subsidiaries.

The purpose of connecting loans and capital to allow bank lending to load on the shock, dZ_t (see also Klimenko et al. (2016) and Coimbra and Rey (2023)). As a reminder, a fraction $\delta dt - \sigma dZ_t$ of capital is destroyed over dt, which is also the fraction of loans that default. Therefore,

$$[1 - \underbrace{(\delta dt - \sigma dZ_t)}_{\textit{default fraction}}] \qquad \underbrace{(1 + R_t dt)}_{\textit{principal \& interest repaid}} \qquad = \; 1 + R_t dt - (\delta dt - \sigma dZ_t) \tag{4}$$

is the banker's return on loans. Note that high-order infinitesimal terms of magnitude below dt are ignored. Negative shocks, $dZ_t < 0$, increase the fraction of loans in default, reducing the return.

Bankers can issue risk-free bonds to households at t, which mature at $t + dt$ with an interest rate $r_t dt$, where r_t is to be determined in equilibrium.[18] As previously discussed, a representative banker is endowed with wealth—bank equity capital—equal to e_0 at $t = 0$. Let x_t denote a banker's leverage, i.e., the ratio of lending (assets) to equity. The law of motion of bank equity, e_t, is

$$\begin{aligned} de_t = & \underbrace{e_t x_t}_{\textit{loan value}} \; \underbrace{[R_t dt - (\delta dt - \sigma dZ_t)]}_{\textit{net loan return}} - \underbrace{e_t (x_t - 1)}_{\textit{debt value}} \; \underbrace{r_t dt}_{\textit{debt cost}} \; - \; \underbrace{dc_t^B}_{\textit{payout / issuance}} \\ & = e_t [r_t + x_t (R_t - \delta - r_t)] \, dt + e_t x_t \sigma dZ_t - dc_t^B \, . \end{aligned} \tag{5}$$

When raising equity from households ($dc_t^B < 0$), bankers face a proportional issuance cost χ.[19] As a result, bankers have the tendency to perpetually postpone payout (or consumption) and save out of the financial constraint, which in turn causes their wealth to outgrow the rest of economy. Following the literature (DeMarzo and Sannikov, 2006; Brunnermeier and Sannikov,

[17] Under $l_t > q_t^K$, the entrepreneur would default on the loan, give up capital to the lending bank, and abscond l_t.

[18] The short-term debts are safe, because under diffusive (Brownian) shocks, banks may continuously adjust balance sheets to avoid negative equity. It will be shown that when equity is sufficiently low, banks optimally raise equity.

[19] To raise one dollar, a banker has to give $1+\chi$ worth of equity to investors. This dilution cost captures informational frictions in Myers and Majluf (1984) and Dittmar and Thakor (2007). Here negative payout, $dc_t^B < 0$, corresponds to equity issuance (Brunnermeier and Sannikov, 2014; Klimenko et al., 2016; Phelan, 2016; Van den Heuvel, 2002)).

2014), I assume that the discount rate of the bankers, denoted by $\overline{\rho}$, is greater than entrepreneurs' and households', i.e., $\overline{\rho} > \rho$, so that the bankers would consume if their equity capital is sufficiently high. Let $\mathbb{I}_{\{.\}}$ be an event indicator. A representative banker's objective function is given by

$$\mathbb{E}\left[\int_{t=0}^{\infty} e^{-\overline{\rho}t}\left(1+\chi\mathbb{I}_{\{dc_t^B<0\}}\right)dc_t^B\right], \tag{6}$$

where, when issuing equity (i.e., $dc_t^B < 0$), the banker faces a loss of χdc_t^B from the issuance cost. Imposing the equity issuance cost is one way to model intermediaries' financial constraints. Alternative approaches in the literature achieve the same purpose of limiting balance-sheet capacity.[20]

Let E_t denote the aggregate equity of representative bankers, $\int_{n\in\mathbb{B}} e_t(n)dn$. The credit-market clearing condition equates aggregate bank lending, $E_t x_t$, to entrepreneurs' borrowing, $K_t l_t$:

$$E_t x_t = K_t l_t. \tag{7}$$

Because bank credit is essential for financing investment, we have investment per unit of capital $i_t = l_t$. Equilibrium analysis describes the credit-market dynamics, for example, how l_t is determined and its cyclical properties. In summary, the economy's productive capital grow through entrepreneurs' investments funded by bank loans. Banks issue debt and equity to households.

3 Equilibrium

The model has two frictions. On the credit-demand side, entrepreneurs' borrowing is subject to a pledgeability constraint (3). On the supply side, credit is channeled by bankers whose balance-sheet capacity is constrained due to the equity issuance cost. Before equilibrium analysis, Section 3.1 describes a benchmark setting where households bypass bankers and directly lend to entrepreneurs and entrepreneurs commit to repay and thus are not subject to the borrowing constraint.

[20]He and Krishnamurthy (2012) emphasize principal-agent frictions and the resultant financial constraints.

3.1 The Frictionless Benchmark

At time t, an entrepreneur chooses the amount to borrow and invest, taking as given the equilibrium value of capital, q_t^K. Her investment is successful with probability λdt: at the margin, one unit of goods invested turns into $F'(l_t)$ units of capital, creating a net marginal gain of $q_t^K F'(l_t) - 1$. Therefore, the expected marginal return on investment is $\left[q_t^K F'(l_t) - 1\right] \lambda dt$. The expected cost of credit is $(R_t - \delta)dt$. A reduction δdt appears in the borrowing cost because capital as perishes at an expected rate of δdt, allowing the entrepreneur to default on the corresponding loans.

Lemma 1 (Credit Demand) *Entrepreneurs' demand for bank credit within the investment limit (i.e., $l_t < \bar{l}$) satisfies the following optimality condition that equates the expected cost of borrowing and expected marginal profits from investment:*

$$R_t - \delta = \lambda \left[q_t^K F'(l_t) - 1\right] . \tag{8}$$

The proof of Lemma 1 is in Appendix A. In equilibrium, the expected return from lending to entrepreneurs, $R_t - \delta$, is equal to ρ, the households' required rate of return (i.e., their discount rate). Replacing the borrowing cost in Equation (8) by ρ, I obtain the following equilibrium condition:

$$\lambda \left[q_t^K F'(l_t) - 1\right] = \rho. \tag{9}$$

Fully solving the equilibrium requires solving the endogenous value of q_t^K. First, we conjecture that q_t^K has the following law of motion in equilibrium:

$$\frac{dq_t^K}{q_t^K} = \mu_t^K dt + \sigma_t^K dZ_t, \tag{10}$$

where the drift term μ_t^K and the diffusion term σ_t^K are endogenous and determined in equilibrium.

Lemma 2 (Capital Valuation) *The value of capital, q_t^K, satisfies the equilibrium condition:*

$$q_t^K = \frac{\alpha + \lambda \left[q_t^K F(l_t) - l_t\right] - (R_t - \delta) l_t}{\rho - (\mu_t^K - \delta + \sigma \sigma_t^K)} . \tag{11}$$

Equation (11) resembles the Gordon growth formula. The proof of Lemma 2 is in Appendix A. The numerator reflects the fact that capital is both directly productive and endows entrepreneurs with growth opportunities. Investing in growth requires credit, so one unit of capital is paired with l_t units of loans with an effective cost of $R_t - \delta$. The denominator contains the discount rate, ρ, and the growth rate, which in turn can be decomposed into the expected change in the capital value, μ_t^K, the expected destruction rate, δ, and Itô's quadratic covariation between stochastic change of the unit value of capital, q_t^K, and stochastic quantity via capital destruction, $\sigma_t^K \sigma$.

The equilibrium values of credit and capital, denoted respectively by l_{FB} and q_{FB}^K (where "FB" is for this "first-best"from this frictionless economy), are jointly determined as follows.

Proposition 1 (Frictionless Equilibrium) *In the frictionless economy, the equilibrium value of capital is given by*

$$q_{FB}^K = \frac{\alpha + \lambda \left[q_{FB}^K F\left(l_{FB}\right) - l_{FB} \right] - \rho l_{FB}}{\rho + \delta} \,. \tag{12}$$

Let l_{FB}^* denote the level of bank credit per unit of capital that solves the optimality condition (9)

$$\lambda \left[q_{FB}^K F'\left(l_{FB}^*\right) - 1 \right] = \rho \,, \tag{13}$$

so we have the equilibrium level of credit given by $l_{FB} < l_{FB}^*$ if $l_{FB}^* < \bar{l}$; otherwise $l_{FB} = \bar{l}$.

Equations (13) and (12) are, respectively, equations (9) and (11) with μ_t^K and σ_t^K equal to zero under a constant q_t^K in equilibrium. Proposition 1 establishes the benchmark (first-best) levels of credit and capital value that can be later compared against the values under financial frictions. The proof is in Appendix A. One key message from the proposition is that, in the frictionless economy, the level of credit provision is constant—that is, credit cycles do not exist.

In the following analysis, I impose the parameter condition

$$\bar{l} > q_{FB}^K,$$

which implies that the pledgeability constraint, $l_t \leq q_t^K$, binds before the constraint of investment capacity, $l_t \leq \bar{l}$. Therefore, the relevant friction on the credit demand side is the pledgeability constraint rather than the technological constraint on the size of investment.

3.2 Credit Supply and Demand Curves

I analyze the equilibrium where credit supply requires bankers' intermediation. There are two frictions, the pledgeability constraint on entrepreneurs' borrowing and bankers' equity issuance cost. I will derive the credit supply and demand curves and characterize how they evolve with the state variable that is the ratio of bank equity to the stock of entrepreneurs' productive capital,

$$\eta_t \equiv \frac{E_t}{K_t}. \tag{14}$$

This ratio measures the scarcity of intermediation capital (bank equity) relative to the size of real economy. The following proposition states that η_t is a key state variable in the Markov equilibrium.

Proposition 2 (Markov Equilibrium) *For any endowments* $\{k_0(n), n \in \mathbb{E}\}$ *and* $\{e_0(n), n \in \mathbb{B}\}$, *there exists a Markov equilibrium with* η_t *as the state variable.*

The proof is in Appendix A. In the following, to differentiate demand and supply, let l_t^S denote bankers' supply of credit, l_t^D denote entrepreneurs' demand for credit, and l_t denote the market-clearing level of credit per unit of capital. In the frictionless economy, credit supply is perfectly elastic at the loan rate $R_t = \rho + \delta$, where ρ is the households' discount rate and δ is the expected default rate. When financially constrained banks intermediate credit supply, the loan rate deviates from this frictionless benchmark, and credit supply is no longer perfectly elastic.

Credit supply. A banker's decision to consume ($dc_t^B > 0$) or to raise equity ($dc_t^B < 0$) depends on the marginal value of equity. The homogeneity property of banker's problem suggests that value function is linear in equity, $q_t^B e_t$.[21] In equilibrium, the marginal value of equity cannot fall below one, in which case the banker is better off consuming one unit of goods than retaining equity. In equilibrium, q_t^B also cannot rise above $1 + \chi$, in which case, paying the issuance cost and raising equity is strictly profitable. Therefore, we have q_t^B bounded in the interval $[1, 1 + \chi]$. If equity issuance is frictionless ($\chi = 0$), the banker is no longer financial constrained, and $q_t^B = 1$ in all

[21] This conjecture of value function is formally proven in the appendix where it is shown that this functional form is consistent with the Hamilton-Jacobi-Bellman (HJB) equation.

states of the world. The next lemma shows that q_t^B reflects the scarcity of bank equity capital relative to the size of production sector that demands bank credit. The proof is in Appendix A.

Lemma 3 (Bank Equity Scarcity) *The marginal value of bank equity, $q_t^B = q^B(\eta_t)$, is a monotone function of η_t, and η_t, is bounded in $\left[\underline{\eta}, \overline{\eta}\right]$ where $\underline{\eta}$ and $\overline{\eta}$, are endogenously determined by the optimality conditions for bank equity issuance and payout (consumption), respectively,*

$$q^B\left(\underline{\eta}\right) = 1 + \chi\,, \ \textit{and}\ \ q^B\left(\overline{\eta}\right) = 1\,. \tag{15}$$

At any $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right)$, $q^B(\eta_t)$ is strictly decreasing in η_t, i.e., $\frac{dq^B(\eta_t)}{d\eta_t} < 0$, and, at $\eta_t = \overline{\eta}$, $\frac{dq^B(\eta_t)}{d\eta_t} = 0$.

The marginal value of equity, $q_t^B = q^B(\eta_t)$, is highest, equal to $1+\chi$ at $\underline{\eta}$, the equity issuance (lower) boundary of η_t where bank equity capital is the most scarce. At the other extreme, when η_t rises to $\overline{\eta}$, the bank is sufficiently well-capitalized, so $q_t^B = 1$ and it is optimal to consume as retaining equity no longer adds value.[22] The ratio of bank equity to capital stock, η_t, is the state variable of the economy, reflecting the scarcity of bank equity and driving its marginal value.

By Itô's lemma, η_t has the following law of motion:

$$\frac{d\eta_t}{\eta_t} = \underbrace{\left\{r_t + x_t\left(R_t - \delta - r_t\right) - \left[\lambda F\left(l_t\right) - \delta\right] - \left(x_t - 1\right)\sigma^2\right\}}_{\triangleq\ \mu_t^\eta} dt + \underbrace{\left(x_t - 1\right)\sigma}_{\triangleq\ \sigma_t^\eta} dZ_t - dy_t\,, \tag{16}$$

where $dy_t = dc_t^B/e_t$ denotes bankers' payout- or issuance-to-equity ratio. Its drift is determined by the expected growth of bank equity (see (5)) relative to that of productive capital.[23] The diffusion term, $\left(x_t - 1\right)\sigma$, is positive because, in equilibrium, bank leverage (asset-to-equity ratio) x_t is greater than 1. Therefore, η_t loads positively on the aggregate shock, $\sigma_t^\eta > 0$.[24] Under $\sigma_t^\eta > 0$, bank equity becomes more scarce—η_t decreases and drives q_t^B higher—after negative shocks ($dZ_t < 0$). As a reminder, negative shocks destroy more capital and increase the number of delinquent loans.

[22] The amounts of issuance at $\underline{\eta}$ and payout at $\overline{\eta}$ exactly offset variation in η_t that would otherwise cause η_t to move beyond the two reflecting boundaries $\underline{\eta}$ and $\overline{\eta}$. The proof of Lemma 3 provides further discussion on the boundaries.

[23] The term $\left(x_t - 1\right)\sigma^2$ in the drift is from Itô's quadratic covariation given that $dK_t = F(l_t)\lambda dt - \left(\delta dt - \sigma dZ_t\right)K_t$.

[24] The model generates cyclicality in bank equity management that is consistent with the evidence (Baron, 2020; Adrian, Boyarchenko, and Shin, 2015): following positive shocks, η_t increases and approaches the payout boundary, $\overline{\eta}$, while negative shocks pushes η_t closer to the equity issuance boundary, $\underline{\eta}$.

Since more loans default when the marginal value of equity, q_t^B, is higher, bankers charge a risk premium when lending to entrepreneurs, akin to the asset pricing models where investors require excess expected return for holing assets that deliver low returns when the marginal value of wealth is high. In continuous-time settings, the risk premium is the negative of instantaneous covariance between asset return and the growth rate of marginal value of wealth (e.g., Cochrane, 2005). The same logic holds here except that loan risk premium emerges from financial friction rather than risk aversion. Let ϵ_t^B denote the elasticity of marginal value of bank equity, q_t^B, to η_t:

$$\epsilon^B(\eta_t) = \frac{dq^B(\eta_t)/q^B(\eta_t)}{d\eta_t/\eta_t} < 0 \,. \tag{17}$$

for $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right)$, and, at $\eta_t = \overline{\eta}$, $\epsilon^B(\overline{\eta}) = 0$ according to Lemma 3. By Itô's lemma, the shock loading of the growth rate of q_t^B (i.e., dq_t^B/q_t^B) is $\epsilon_t^B \sigma_t^\eta$. The return on loans has a shock loading σ (see (4)). Therefore, the loan risk premium is $-\epsilon_t^B \sigma_t^\eta \sigma$, i.e., the negative of instantaneous covariance between the growth rate of q_t^B, the marginal value of bank equity, and return on loans. A formal proof is provided in Appendix A. The next lemma summaries the results on loan risk premium.

Lemma 4 (Loan Risk Premium) *The equilibrium loan rate, R_t, is given by*

$$R_t = r_t + \delta + \gamma_t^B \sigma, \tag{18}$$

where the loan risk premium can be decomposed into the quantity of loan risk, σ, and the bankers' price of risk, γ_t^B, that is in turn given by

$$\gamma_t^B = -\epsilon^B(\eta_t)\sigma_t^\eta = -\epsilon^B(\eta_t)\,(x_t - 1)\,\sigma > 0 \,. \tag{19}$$

The net interest margin, i.e., $R_t - r_t$, compensates both the expected default rate, δ, and loan risk premium, $\gamma_t^B \sigma$.[25] As previously discussed, under $\chi = 0$, the marginal value of bank equity is a constant, i.e., $q_t^B = 1$, and thus does not load on shocks (i.e., $\epsilon_t^B = 0$). As a result, loan risk premium is zero, and the net interest margin only contains the expected default rate, δ. In this

[25]This result is closely related to the "credit spread puzzle" that less than one-half of the variation in credit spreads can be attributed to borrowers' expected default probability (i.e., δ in the model). The unexplained portion is due to time-varying risk premium (e.g., Collin-Dufresn, Goldstein, and Martin, 2001).

case, the loan rate, $R_t = \rho + \delta$, equal to the loan rate in the frictionless benchmark economy in Section 3.1. Under $\chi > 0$, the loan rate is above $\rho + \delta$. Therefore, in the model, loan risk premium ultimately emerges from bankers' financial frictions (i.e., the equity issuance cost).

Next, I derive the credit supply curve. First, I pin down bankers' debt cost. Bankers can issue debt to households at interest rate $r_t = \rho$, where ρ is households' discount rate.

Lemma 5 (Equilibrium Risk-free Rate) *The interest rate offered by bank debt, r_t, is equal to ρ.*

Lemmas 3, 4, and 5 provide all the necessary ingredients for deriving the credit supply curve. Credit supplied per unit of capital, l_t^S, is equal to aggregate bank lending, $x_t E_t$, divided by the aggregate stock of capital, K_t, i.e., $l_t^S = x_t E_t / K_t = x_t \eta_t$. Substituting this expression into (19) and substituting r_t with its equilibrium value ρ, I obtain the credit supply curve from (18):

$$R_t = \rho + \delta + b_t \left(l_t^S - \eta_t \right) , \tag{20}$$

where the last term is from the risk premium $\gamma_t^B \sigma$, and the slope, $b_t = b(\eta_t)$, is a function of η_t:

$$b(\eta_t) = -\frac{\epsilon^B(\eta_t)\sigma^2}{\eta_t} . \tag{21}$$

As the ratio of bank equity to capital stock, η_t, evolves over time, credit-supply elasticity varies. The next proposition characterizes a credit supply curve. The proof is in Appendix A.

Proposition 3 (Credit Supply Curve) *The credit supply curve in the space of loan rate and loan amount, (R_t, l_t^S), given by (20) is upward-sloping, i.e., $b(\eta_t) > 0$, when η_t is in $[\underline{\eta}, \overline{\eta})$, and $b(\overline{\eta}) = 0$.*

Credit demand. The next proposition extends Lemma 1 on entrepreneurs' credit demand by incorporating the pledgeability constraint and emphasizes that capital value, q_t^K, in equilibrium is no longer a constant as in the frictionless economy but instead it varies with η_t.

Proposition 4 (Credit Demand Curve) *The credit demand curve is characterized as follows:*

- *Given q_t^K, the entrepreneurs' optimality condition (8) for l_t^D holds and implies a downward-sloping demand curve in the space of loan rate and loan amount, (R_t, l_t^D), i.e., $\frac{\partial l_t^D}{\partial R_t} \leq 0$, if the pledgeability constraint does not bind, i.e., $l_t^D < q_t^K$; otherwise, $l_t^D = q_t^K$.*

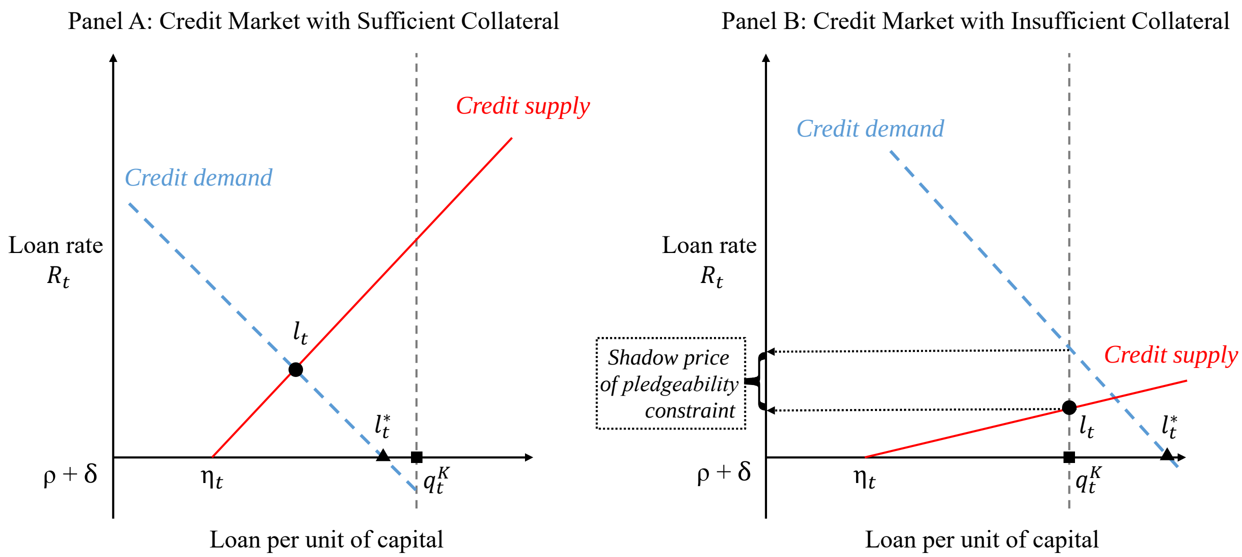

Figure 1: **Credit Market.** In the space of credit per unit of capital, l_t (the horizontal axis), and loan rate, R_t (the vertical axis), this figure plots the entrepreneurs' credit demand (blue dashed line), equation (8), the pledgeability constraint (3) on l_t at q_t^K marked by the square point (vertical grey dashed line), and bankers' credit supply (red solid line), equation (20), given η_t, q_t^K and ϵ_t^B. The credit market clears at the intersection marked by the round points. The triangle point marks the investment target given by equation (22). Panel A illustrates the case where the pledgeability constraint does not bind, i.e., $l_t < q_t^K$. Panel B illustrates the case where the pledgeability constraint binds, $l_t = q_t^K$.

- *Capital value, $q_t^K = q^K(\eta_t)$, is a function of η_t. It shifts the credit demand curve in the space of (R_t, l_t^D): given any loan rate R_t, credit demand is increasing in q_t^K, i.e., $\frac{\partial l_t^D}{\partial q_t^K} \geq 0$.*

The proof is in Appendix A. Given q_t^K, the concavity of $F(\cdot)$ implies a downward-sloping curve in the space of loan rate and loan amount, (R_t, l_t^D) through the optimality condition (8). The value of capital, q_t^K, reflects the entrepreneurs' expectations of capital growth path and cost of credit, shown by the equilibrium condition (11). An increase of q_t^K reflects an improved expectation of capital growth and future credit conditions, so entrepreneurs want to obtain more credit to finance the creation of new capital, resulting in the credit demand curve shifting outward, i.e., $\frac{\partial l_t^D}{\partial q_t^K} \geq 0$ for any value of R_t, in the region where the pledgeability constraint does not bind.

Credit market clearing. Figure 1 illustrates a time-t snapshot of the credit market. Given the value of η_t, capital value and the elasticity of margin value of bank equity, i.e., $q_t^K = q^K(\eta_t)$ and $\epsilon_t^B = \epsilon^B(\eta_t)$ respectively, are pinned down so that we can locate the credit demand and supply curves in the space of loan rate (the vertical axis) and loan amount per unit of capital (the horizontal axis). In both Panel A and B, the credit supply curve (red solid line) is drawn starting from $l_t^S = \eta_t$, where bank lending is fully equity-financed (i.e., the asset-to-equity ratio, $x_t = 1$). At $l_t^S = \eta_t$, the loan rate is $\rho + \delta$ as shown in (20), and as l_t^S increases, the loan rate increases with the slope given by $b(\eta_t)$ in Proposition 3. In both Panel A and B of Figure 1, the credit demand curve has two parts. The gray dotted line represents the pledgeability constraint and located at $l_t^D = q_t^K$. The blue dashed line represents the relationship between loan rate and loan amount when the pledgeability constraint is not binding, given by the optimality condition (8). In the figure, l_t^* denotes where this line crosses the hypothetical loan rate $R_t = \rho + \delta$, i.e., the level of borrowing when the loan rate does not incorporate loan risk premium and when the pledgeability constraint does not bind:

$$\lambda \left[q_t^K F'\left(l_t^*\right) - 1\right] = \rho. \tag{22}$$

The market-clearing (l_t, R_t) is marked by the round points in Figure 1.

In Panel A, the pledgeability constraint does not bind, and the market clears at $l_t < q_t^K$, where the demand and supply curves implied by (8) and (20), respectively, intersect:

$$\lambda \left[q^K(\eta_t) F'\left(l_t\right) - 1\right] = \rho + b(\eta_t)\left(l_t - \eta_t\right) . \tag{23}$$

In Panel B, the pledgeability constraint binds, $l_t = q_t^K$. The shadow price of pledgeability constraint is equal to the wedge between market-clearing R_t and the hypothetical (higher) loan rate at the intersection of demand and supply curves implied by (8) and (20), i.e., the hypothetical market-clearing rate if the pledgeability constraint were to be eliminated. Under the equilibrium loan rate R_t, entrepreneurs would like to borrow more, as the marginal value from investing, $\lambda[q_t^K F'(l_t) - 1]$, is above the borrowing cost, $R_t - \delta$, i.e., the right side of (8) is greater than the left side.

The dynamic economy can be viewed as a sequence of credit-market clearing that progresses over time, driven by η_t. The state variable, η_t, dictates the credit-market conditions. It enters the credit demand curve via $q^K\left(\eta_t\right)$ in the pledgeability constraint (3) and optimality condition (8). It enters the credit supply curve given by (20) through $\epsilon^B\left(\eta_t\right)$ in slope of the supply curve, $b(\eta_t)$.

3.3 Credit Market Dynamics

The analysis so far focuses on credit-market clearing in one state of the world given by the value of η_t. Once η_t is fixed, the values of $q^K(\eta_t)$ and $\epsilon^B(\eta_t)$ are given, and they determine the locale and shape of credit demand and supply curves, respectively. Next, I characterize how the credit demand and supply curves evolve as η_t varies over time and responds to shocks.

A parameter condition is imposed for the next proposition: $\frac{2(\bar{\rho}-\rho)}{\sigma^2} > 1$. Bankers' discount rate being higher than households' (impatience) is a force contributing to leverage (i.e., bankers borrowing from households). The condition requires discount-rate wedge to be large enough after loan risk adjustment so that bank leverage, x_t, is high enough.[26] In Section 3.4, numerical solutions show realistic leverage under this condition. The proof of next proposition in Appendix A.

Proposition 5 (The Procyclicality of Credit Supply) *The slope of credit supply curve, $b(\eta_t)$ given by (20), is strictly decreasing in η_t, i.e., $\frac{d\,b(\eta_t)}{d\eta_t} < 0$. Therefore, an increase of η_t causes credit supply, given by Proposition 3, to expand at any loan rate, R_t.*

Credit supply becomes more elastic when banks become better capitalized. Graphically, an increase from η_t to η_{t+dt} rotates the credit supply curve clockwise by reducing the slope according to Proposition 5 and moves the intercept outward as shown in (20). This is illustrated by the credit-supply expansion from Line 1 to Line 2 in Panel A of Figure 2. So, for any given R_t, bankers lend more when η_t increases. As a result, the market-clearing point moves A to B along the credit demand curve. State-dependent credit supply emerges from bankers' financial constraint in the form of equity issuance cost. As previously discussed, if $\chi = 0$, $q_t^B = 1$ and $b_t = 0$ (under $\epsilon_t^B = 0$), and credit supply is always perfectly elastic at $R_t = \rho + \delta$. Under $\chi > 0$, credit supply is more elastic when banks are better capitalized and thus less financially constrained. The illustration in Panel A holds the credit demand curve fixed. However, in equilibrium, as η_t increases and the credit-supply condition improves, the credit demand curve responds as well, illustrated in Panel B. I will explain the mechanism following the next proposition. The proof is in Appendix A.

[26] This condition is satisfied under standard parameter values. For example, for $\sigma = 0.1$ (see Appendix C for parameters used in the numerical examples), the condition requires a discount-rate wedge of at least 50 basis points.

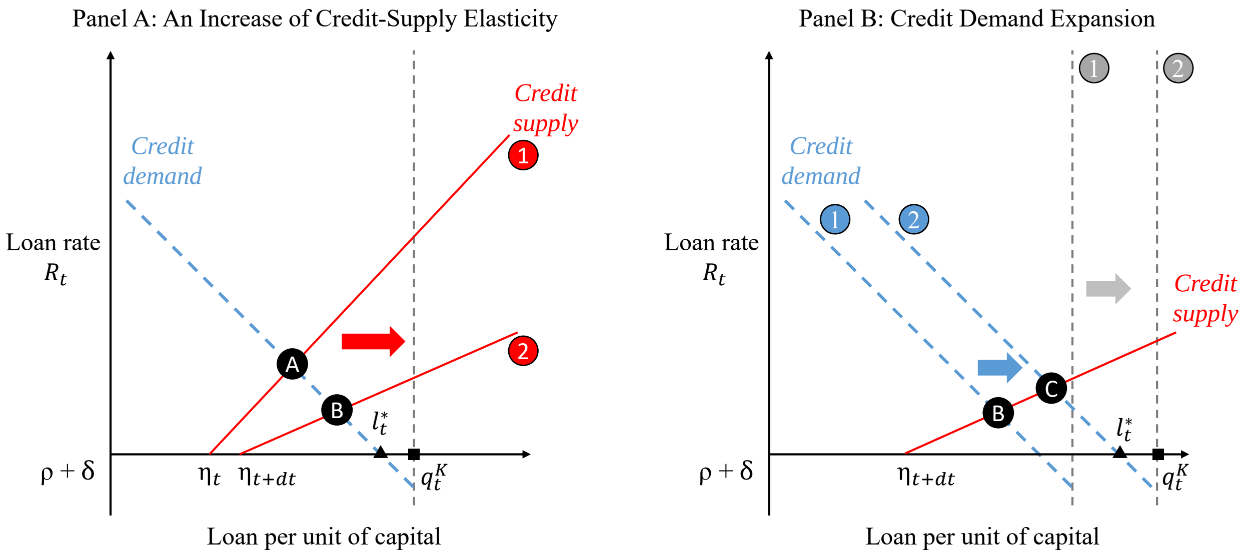

Figure 2: **Credit Market Dynamics.** In the space of credit per unit of capital, l_t (the horizontal axis), and loan rate, R_t (the vertical axis), this figure plots the entrepreneurs' credit demand (blue dashed line), equation (8), the pledgeability constraint (3) on l_t at q_t^K marked by the square point (vertical grey dashed line), and bankers' credit supply (red solid line), equation (20), given η_t, q_t^K and ϵ_t^B. The credit market clears at the intersection marked by the black round points. The triangle point marks the credit level given by equation (22). Panel A illustrates the expansion of credit supply through the decline of b_t in Proposition 5, from line 1 to 2, which is induced by the increase of the state variable η_t to η_{t+dt}. The resultant movement of market-clearing point is from point A to B. Panel B illustrates the expansion of credit demand that is due to the increase of q_t^K when η_t increases in η_{t+dt} (Proposition 6). The increase of q_t^K causes the blue dashed line (given by Equation (8)) and the vertical gray dashed line (pledgeability constraint) to shift outward, from position 1 to 2. The resultant movement of market-clearing point is from point B to C.

Proposition 6 (The Procyclicality of Credit Demand) *The equilibrium capital value, $q^K(\eta_t)$, is strictly increasing in η_t, i.e., $\frac{dq^K(\eta_t)}{d\eta_t} > 0$, in $\left(\underline{\eta}, \overline{\eta}\right)$. When $\eta = \underline{\eta}$ or $\overline{\eta}$, $\frac{dq^K(\eta_t)}{d\eta_t} = 0$. Therefore, an increase of η_t causes credit demand, given by Proposition 4, to expand at any loan rate, R_t.*

The procyclicality of credit demand arises from contemporaneous and intertemporal channels illustrated by Figure 3. Consider $\eta_t \in \left(\underline{\eta}, \overline{\eta}\right)$ and a positive shock, dZ_t, increases η_t. Credit supply is more elastic and banks charge a lower loan rate (Proposition 5). As a result, entrepreneurs borrow more and grow capital faster, which increases capital value, as shown in (11). The increase of q_t^K implies greater gains from investment and also relaxes the pledgeability constraint (black arrows), inducing a stronger credit demand (teal blue arrows). The market-clearing level of credit rises because both credit supply and demand expand. This is the *contemporaneous* channel.

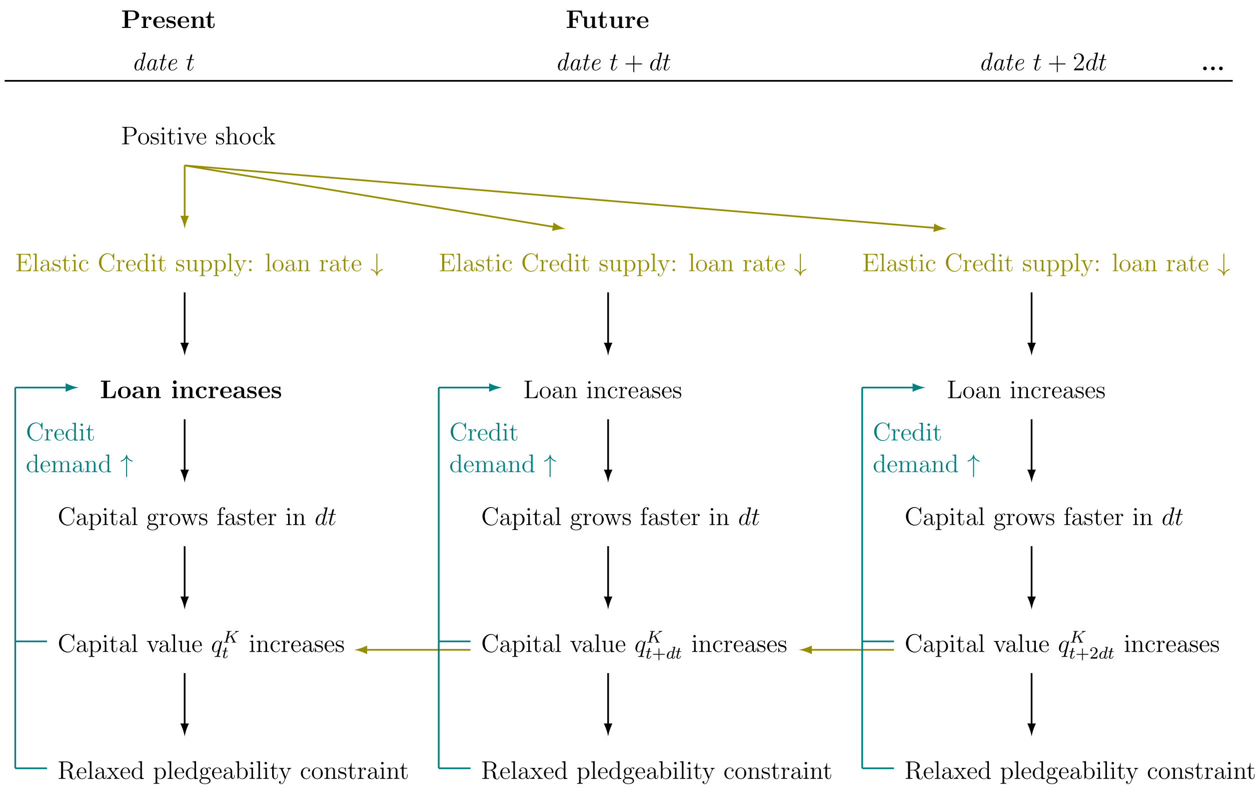

Figure 3: **Credit Market Procyclicality: Contemporaneous and Intertemporal Feedback Channels.** This figure illustrates the mechanisms behind procyclicality of the credit market. Following positive shocks, the supply curve becomes more elastic and bankers charge a lower loan rate, which leads to more borrowing by entrepreneurs, faster capital growth, and higher capital value (black arrows). Higher capital value relaxes the pledgeability constraint, allowing entrepreneurs to borrow even more (black arrow). As the shock's impact is persistent, the path of expected credit cost shifts downward and path of expected capital value shifts upward. The expected increase of capital value feeds into a higher current value of capital (olive arrows), which in turn stimulates entrepreneurs' current investment need and expands the credit demand, which further raises the equilibrium level of credit (teal blue arrows).

An *intertemporal* channel further increases capital value because the shock's impact is persistent. Financial constraint (the equity issuance cost) leads to precautionary savings—that is, following the positive shock, bankers would not immediately consume the associated earnings but instead retain earnings to accumulate equity, i.e., the financial slack, to prevent costly equity raising when η_t is low. As a result of retained earnings, not only the current value of η_t increases, the positive shock lifts the whole path of η_t into the future. Therefore, the positive shock not only leads to a credit-supply expansion now but is also anticipated to push outward future credit supply curves.[27] Accordingly, entrepreneurs foresee a higher market-clearing level of credit provision on the equilibrium path and faster capital growth rates going forward. An improved growth trajectory feeds into a higher current value of capital, q_t^K, through the expected appreciation μ_t^K, as shown in (11). Moreover, higher values of q_t^K relax the pledgeability constraint along the growth path. The olive-colored arrows in Figure 3 show the intertemporal channel of such credit-demand expansion.

Proposition 6 shows that capital value, $q_t^K = q^K(\eta_t)$, is an increasing function of state variable η_t, the ratio of bank equity to capital stock that measures the intermediation capacity relative to the size of economy. Figure 3 demonstrates the mechanisms behind $\frac{dq^K(\eta_t)}{d\eta_t} > 0$.[28] It is ultimately the current credit-supply conditions and entrepreneurs' expectation of future credit-supply conditions that drive capital value, which in turn determines the credit demand curve. In other words, procyclicality originates in the banking sector. The direct impact of shocks is on bank equity and its intermediation capacity. Credit demand also responds because the value of capital reflects entrepreneurs' expectation of credit provision and capital growth on the equilibrium path.

Back in Figure 2, Panel A shows the credit supply curve shifts outward and becomes more elastic following an increase of η_t (Propositions 5) and the market-clearing point moves from Circle A to B. The mechanism does not end here. In Panel B, q_t^K increases under a higher η_t (Proposition 6) and causes the demand curve (blue dashed line given by (8)) to move from line 1 to 2 and pushes outward the vertical gray dashed line that marks the pledgeability constraint $l_t \leq q_t^K$. The market-clearing point moves further from Circle B to C along the more elastic supply curve.

The next proposition summarizes how the equilibrium level of credit provision varies with η_t. The proof is in Appendix A. In the frictionless economy, the equilibrium level of credit provision, l_t, is constant, and there does not exist credit cycle. In contrast, when financially constrained banks

[27] Mathematically, in the interior region, i.e., $\eta_t \in (\underline{\eta}, \overline{\eta})$, $q^B(\eta_t)$ is strictly above 1 under $\chi > 0$ (i.e, there exists equity issuance costs). Given that $q^B(\eta_t)$ is a continuous function of η_t, q_t^B cannot suddenly jump downward to 1 and trigger payout following a positive Brownian (diffusive) shock to η_t, i.e., $dZ_t > 0$, that cannot cause η_t to jump from the interior region to the payout (upper) boundary, $\overline{\eta}$. Therefore, the increase of η_t is expected to dissipate only gradually into the future as a subset of future realized paths of η_t lead to $\overline{\eta}$ after η_t increases over certain periods of time. In contrast, under $\chi = 1$, $q^B(\eta_t) = 1$ and bankers consume any earnings and do not precautionarily save. Therefore, the persistent impact of shocks is ultimately due to the equity issuance cost.

[28] Asset price variations tend to be driven by discount-rate variations (e.g. Cochrane (2011)). In the model, discount rate is fixed at ρ, so the variation in q_t^K is purely driven by entrepreneurs' investment gains and borrowing cost.

are required to intermediate the supply of funding, credit cycle emerges, that is l_t varies with η_t.

Proposition 7 (The Procyclicality of Equilibrium Credit) *The equilibrium level of credit per unit of capital, $l_t = l(\eta_t)$, is strictly increasing in η_t, i.e., $\frac{dl(\eta_t)}{d\eta_t} > 0$.*

Next, I characterize the sources of inefficiency. In Figure 1, the level of credit, l_t^*, which is given by (22) and marked by the triangular points in Figure 1, acts as an efficiency benchmark. It is the market-clearing level of credit in the absence of entrepreneurs' pledgeability constraint and loan risk premium that is due to bankers' equity issuance cost. In Panel A, the wedge between the equilibrium level, l_t, and l_t^* is caused by loan risk premium. Should bankers' equity issuance cost is zero ($\chi = 0$), the marginal value of bank equity would be a constant equal to one ($q_t^B = 1$ and thus $\epsilon^B(\eta_t) = 0$), and credit supply would be perfect elastic at the loan rate $R_t = \rho + \delta$. Therefore, the wedge between l_t and l_t^* in Panel A is ultimately due to bankers' equity issuance cost. In Panel B, $l_t < l_t^*$ is due to the binding pledgeability constraint, $l_t = q_t^K$. The frictions—bankers' equity issuance cost and entrepreneurs' pledgeability constraint—lead to inefficiency.

The analysis so far has compared the equilibrium level of credit, l_t, against l_t^*; yet, l_t^* still depends on capital value q_t^K (see the definition of l_t^* given by (22)), and q_t^K is endogenous and reflects entrepreneurs' expectation of future credit provision and capital growth on the equilibrium path. Therefore, using l_t^* as an efficiency benchmark serves the purpose of illustrating the mechanisms but does not fully account for inefficiency. The next two propositions formally state that the equilibrium capital value, q_t^K, is below the frictionless benchmark, q_{FB}^K in Proposition 1, and the equilibrium credit provision, l_t, is below l_{FB} in Proposition 1. The proofs are in Appendix A.

Proposition 8 (Capital Value Wedge) $q^K(\eta_t) < q_{FB}^K$ *for* $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right)$ *and* $q^K(\overline{\eta}) \leq q_{FB}^K$.

Proposition 9 (Credit Wedge) $l(\eta_t) < l_{FB}$ *for* $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right)$ *and* $l(\overline{\eta}) \leq l_{FB}$.

This proposition states that the frictions—bankers' equity issuance cost and entrepreneurs' pledgeability constraint—cause the equilibrium level of bank credit and entrepreneurs' investment to be inefficiently low. Proposition 7 implies that, when bank equity increases relative to the size of real economy (i.e., $\eta_t = E_t/K_t$ increases), the resultant expansion of credit supply and demand leads to an increase in $l_t = l(\eta_t)$, narrowing the wedge, $l_{FB} - l_t$, in Proposition 9. As η_t evolves over time, the economy features credit booms (high-η_t states) and credit crunches (low-η_t states).

3.4 Credit Boom-Bust Cycles

The equilibrium is driven by η_t, the ratio of bank equity to capital stock, which measures the scarcity of bank equity. As shown in Proposition 5 and 6, both the credit supply and demand curves vary with η_t. As a result, l_t, the equilibrium credit level, is increasing in η_t (Proposition 7).

Therefore, the key to understanding the credit boom-bust cycle is the law of motion of η_t given by (16), and in particular, its diffusion and drift terms. This law of motion governs the transition across different states of the economy. The diffusion term, $(x_t - 1)\sigma$, which determines how η_t responds to shocks, depends on bank leverage x_t. The first component of the drift, $r_t + x_t(R_t - \delta - r_t)$, is the banks' expected return on equity (ROE), which is the risk-free rate (equal to ρ in equilibrium) plus the expected excess return on lending, $R_t - \delta - r_t$, multiplied by bank loan-to-equity ratio (leverage), x_t.[29] Next, I characterize the dynamics of bank leverage and ROE, and then, based on these results, analyze the dynamics of η_t and credit boom-bust cycles. In particular, I will demonstrate that the economy either spends extended periods in credit booms, characterized by high values of η_t and high levels of equilibrium credit provision l_t, or becomes trapped in stagnant credit crunches. The transition between these endogenous regimes often occurs abruptly.

Bank leverage dynamics. To clearly demonstrate the economic forces that drive bank leverage and profitability (ROE), I consider a simple functional form of the investment function: $F(l_t) = \kappa l_t$ if $l_t < \bar{l}$, and $F(l_t) = \kappa \bar{l}$ if $l_t \geq \bar{l}$. Under this functional form, investment is scalable up to $\bar{l}$.[30] And, the following parameter condition is imposed:

$$\lambda \left[\left(\frac{\alpha}{\rho + \delta} \right) \kappa - 1 \right] > \rho.$$

[29] The other components are the expected growth rate of K_t, i.e., $\lambda F(l_t) - \delta$, and the quadratic covariation term.

[30] Note that the pledgeability constraint, $l_t \leq q_t^K$, binds before the capacity constraint, $l_t \leq \bar{l}$, that is $q_t^K < \bar{l}$ because $q_t^K \leq q_{FB}^K$ from Proposition 8 and $q_{FB}^K < \bar{l}$, the parameter condition at the end of Section 2.

Note that $\frac{\alpha}{\rho+\delta}$ is the hypothetical value of capital without growth opportunities, given by the Gorton growth formula with the numerator (cash flow), α, and denominator (discount rate) including the time-discount rate, ρ, and depreciation rate, δ. This condition states that marginal investment is profitable even without future growth opportunities.[31] This condition ensures that investment technology is sufficiently profitable, so what drives credit demand is not technological reasons but entrepreneurs' expectation of future credit-supply conditions, embedded in q_t^K, shown in Figure 3.

Next, I characterize the dynamics of bank leverage and then the dynamics of bank ROE. When the entrepreneurs' pledgeability constraint does not bind, x_t, the ratio of bank asset (loans) to equity, is solved from the credit-market clearing condition (23).[32] When the pledgeability constraint binds, i.e., $l_t = q_t^K$, x_t can be solved as through the market-clearing condition (7) scaled by K_t, i.e., $\eta_t x_t = l_t$, so we obtain $x_t = l_t/\eta_t = q_t^K/\eta_t$. The next lemma summarizes the results.

Lemma 6 (Solving Bank Leverage) *When the entrepreneurs' pledgeability constraint does not bind, bank leverage (asset-to-equity ratio) is given by*

$$x_t = 1 + \left(\frac{\lambda \left[q^K(\eta_t)\kappa - 1 \right] - \rho}{-\epsilon^B(\eta_t)\sigma^2} \right) . \tag{24}$$

When the entrepreneurs' pledgeability constraint binds, bank leverage is given by

$$x_t = \frac{q_t^K}{\eta_t} . \tag{25}$$

When the pledgeability constraint does not bind, the driving force of credit demand is captured by $\lambda \left[q^K(\eta_t)\kappa - 1 \right] - \rho$ in (24), the marginal profit from entrepreneurs' investment. If investment is more profitable, the entrepreneurs' demand for credit strengthens, inducing bankers to raise leverage. Banks' willingness to increase loan risk exposure through leverage also depend on

[31] On the left side of the inequality, investment is successful with probability λdt in creating κ units of no-growth capital (worth $\frac{\alpha}{\rho+\delta}$ per unit) out of 1 unit of goods invested, and with probability $1 - \lambda dt$, the investment fails, acting as a storage technology. On the right side is the required rate of return or discount rate, ρdt. The dt is then canceled on both sides. Proposition A.1 in Appendix A formally proves the equilibrium capital value with growth opportunities, q_t^K, is greater than $\frac{\alpha}{\rho+\delta}$. Therefore, investing in capital with growth opportunities is even more profitable.

[32] Specifically, by substituting out l_t with $x_t\eta_t$ (based on the K_t-scaled market-clearing condition (7)) and substituting out the slope of credit supply curve, $b(\eta_t)$ with $-\epsilon^B(\eta_t)\sigma^2/\eta_t$ (see (21)), x_t can be solved from (23).

their risk-taking capacity, captured by $-\epsilon^B(\eta_t)$, defined in (17), the elasticity of $q^B(\eta_t)$, bankers' marginal value of wealth, with respect to the state variable, η_t. Lemma 3 shows that $q^B(\eta_t)$ is decreasing in η_t, so we have $\epsilon^B(\eta_t) < 0$. The next lemma shows how $\epsilon^B(\eta_t)$ varies with η_t.

Lemma 7 (Bank Risk-Taking Capacity) *$\epsilon^B(\eta_t)$, defined in (17), is strictly increasing in η_t.*

The proof is in Appendix A. When bank equity becomes more scarce (η_t decreases), $\epsilon^B(\eta_t)$ is more negative. As a result, bankers' marginal value of wealth, q_t^B, is more sensitive to variation of the state of economy, and their risk-taking capacity shrinks, reducing leverage x_t. In contrast, as η_t increases, $\epsilon^B(\eta_t)$ becomes less negative, driving up bank leverage. This is the force of procyclical risk-taking capacity on the credit supply (bank) side. Moreover, as shown in Proposition 6, $q_t^K(\eta_t)$ is also increasing in η_t, reflecting the procyclicality of investment profits on the credit demand (entrepreneur) side. Therefore, both the numerator and denominator of x_t in (24) contribute to the procyclicality of bank leverage when the pledgeability constraint does not bind.

In contrast, when the pledgeability constraint binds, bank leverage x_t is countercyclical (decreasing in η_t) as η_t approaches the upper bound, $\overline{\eta}$. As shown in Proposition 6, $dq^K(\eta_t)/d\eta_t$ approaches zero as η_t approaches $\overline{\eta}$ where bankers no longer retain earnings. This is because the intertemporal channel in Figure 3 weakens: as η_t approaches $\overline{\eta}$, entrepreneurs no longer foresee a persistent impact of positive shocks to η_t, because η_t cannot go beyond the bankers' optimal consumption boundary, $\overline{\eta}$ at which any increase in bank equity will not be retained but consumed.

In summary, when firms have spare debt capacity, bank leverage is given by (24) and is procyclical (increasing in η_t). When η_t is sufficiently high, bank leverage becomes countercyclical. The next proposition summarizes the hump-shaped dynamics. The proof is in Appendix A.

Proposition 10 (Bank Leverage Cycle) *When the pledgeability constraint does not bind, $\frac{dx(\eta_t)}{d\eta_t} > 0$. There exists $\widetilde{\eta} < \overline{\eta}$ such that, at $\eta_t > \widetilde{\eta}$, the pledgeability constraint binds and $\frac{dx(\eta_t)}{d\eta_t} < 0$.*

To visualize the bank leverage dynamics in Proposition 10, I solve the Markov equilibrium numerically using the method in Brunnermeier and Sannikov (2014).[33] In Appendix B, I discuss

[33] As in Brunnermeier and Sannikov (2014), solving Markov equilibrium in Proposition 2 involves converting the equilibrium conditions into a system of differential equations for endogenous variables, such as capital value and bankers' marginal value of wealth. The proof of Proposition 2 in Appendix A provides details on the method.

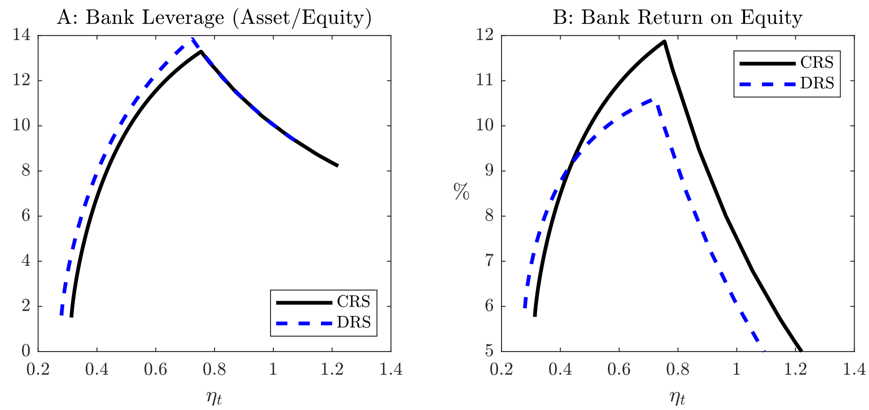

Figure 4: **Bank Leverage and ROE Dynamics.** This figure plots bank leverage and return on equity (ROE) as functions of η_t in Panel A and B, respectively, under CRS (solid line) and DRS (dashed line) investment technologies in the production sector. This figure is based on numerical solutions with parameters calibrated in Appendix B and C.

the calibration of χ, the equity issuance cost by reviewing the related empirical literature. In Appendix C, I discuss the calibration of other parameters. For robustness, I also report the model solution under an alternative investment technology as well. The baseline technology is scalable (constant return-to-scale or "CRS") up to $\bar{l}$, while the dashed line represents a decreasing return-to-scale (DRS) technology, $F(l_t) = \kappa l_t^{\kappa_1}$ with $\kappa_1 < 1$. The calibration exercise in Appendix C intentionally avoids involving any bank characteristics, so the dynamics of bank leverage and ROE can be viewed model predictions rather than calibration targets. Moreover, I emphasize that the model is designed with minimal ingredients to demonstrate theoretical mechanisms rather than to replicate empirical patterns that depend on forces in the model and those outside the model.

Panel A of Figure 4 illustrates how bank leverage, x_t, varies with the state variable, η_t. Consider a path of η_t increasing from $\underline{\eta}$ and eventually reaches $\overline{\eta}$, representing an economy recovering from a credit crunch and evolving into a credit boom. Bank leverage increases at first when the pledgeability constraint does not bind. A contributing factor is the expansion of credit demand fueled by entrepreneurs' expectation of improving credit-supply conditions in the future that in turn contributes to an improving prospect of capital growth as illustrated in Figure 3.

As η_t keeps increasing and passes the threshold $\widetilde{\eta}$, credit supplied by bankers has grown to a level where the entrepreneurs' pledgeability constraint binds (i.e., $l_t = q_t^K$), so, any further growth of equilibrium level of credit solely lies on the rising capital value, q_t^K. However, as η_t approaches

$\overline{\eta}$, the increase of capital value in η_t tapers off, as previously discussed (at $\eta_t = \overline{\eta}$, $\frac{dq^K(\eta_t)}{d\eta_t} = 0$ in Proposition 6). Therefore, the growth of bank asset (lending) cannot catch up with that of equity, resulting debt being "crowded out" and bank leverage, x_t, being countercyclical. Such hump-shared relationship between x_t and η_t holds both CRS and DRS investment technologies.

Finally, I want to highlight an empirical prediction of Proposition 10. In the model, a bank represents a financial intermediary that corresponds to commercial banks or other credit intermediaries in reality. Different types of credit intermediaries serve different borrower clienteles. The model shows that the leverage of credit intermediaries is procyclical when the borrowers' pledgeability constraint does not bind and countercyclical when the pledgeability constraint binds. Therefore, when empirically examining leverage cyclicality of credit intermediaries, it is important to condition on whether their borrowers' pledgeability constraint binds. This observation is relevant for analyzing both the aggregate intermediation sector and the cross-sectional heterogeneity across different types of intermediaries. In reality, it can happen that one type of intermediaries exhibit procyclical leverage as the borrowers they serve do not face a binding pledgeability constraint, while another type of intermediaries exhibit countercyclical leverage as their customers' (borrowers') pledgeability constraint binds. As previously emphasized, the goal of this paper is not to replicate empirically observed patterns of leverage cyclicality, which depend on economic forces in and outside of my model. The goal is to demonstrate new theoretical mechanisms on leverage dynamics and, importantly, their implications on the credit boom-bust cycles.

Bank profitability dynamics. Bank ROE also exhibits a hump-shaped relationship with η_t as bank leverage does. First, consider the case where the pledgeability constraint does not bind:

$$ROE_t = r_t + (R_t - \delta - r_t)\, x_t = \rho + \left(\lambda \left[q_t^K \kappa - 1\right] - \rho\right) x_t \tag{26}$$

where r_t is substituted out by ρ, the equilibrium value, and $\left(\lambda \left[q^K(\eta_t)\kappa - 1\right] - \rho\right)$ replaces $R_t - \delta$ as (8) holds when the pledgeability constraint does not bind. This equation reveals that bank return on equity comes from two sources, entrepreneurs' investment profit that drives credit demand, and leverage, x_t. From Proposition 6, q_t^K is increasing in η_t, and, from Proposition 10, x_t is also increasing in η_t, which together imply that bank ROE is increasing in η_t. When the pledgeability constraint binds (i.e., $\eta_t > \widetilde{\eta}$ where $\widetilde{\eta}$ is defined in Proposition 10), the next proposition shows that bank ROE becomes decreasing in η_t. The proof is in Appendix A.

Proposition 11 (Bank ROE Cycle) *For $\eta_t < \widetilde{\eta}$, where $\widetilde{\eta}$ is defined in Proposition 10, the pledgeability constraint does not bind, and we have $\frac{dROE(\eta_t)}{d\eta_t} > 0$. For $\eta_t > \widetilde{\eta}$, we have $\frac{dROE(\eta_t)}{d\eta_t} < 0$.*

Both bank leverage and return on equity are increasing in η_t (procyclical) when η_t is low and decreasing in η_t (countercyclical) when η_t is high. Such hump-shaped dynamics have important implications on credit cycles. For example, in low-η_t states where the equilibrium level of credit is low (see Proposition 7), both bank ROE and leverage are low. A low ROE implies that the expected growth rate of bank equity and the drift of η_t are low, and a low bank leverage implies that η_t is not very responsive to shocks. As a result, once entering into low-η_t states (credit crunches), the economy gets stuck there. Next, I characterize the credit boom-bust cycles through the law of motion of η_t that in turn depend on the results discussed so far about how bank leverage and ROE.

Credit boom. Bank leverage and ROE dynamics in Proposition 10 and 11, respectively, have critical implications on the long-run dynamics of the economy, i.e., the amount of time the economy spends in credit boom (high-η_t states) and crises (low-η_t states) as described by the stationary distribution of η_t that is formally introduced in the next lemma. The proof is in Appendix A.

Lemma 8 (Stationary Distribution) *The density function of stationary distribution, $p(\eta_t)$, is solved by the following ordinary differential equation (ODE):*

$$\mu^\eta(\eta)\, p(\eta) - \frac{1}{2}\frac{d}{d\eta}\left(\sigma^\eta(\eta)^2\, p(\eta)\right) = 0\,, \tag{27}$$

where $\mu^\eta(\eta)$ and $\sigma^\eta(\eta)$ are defined in the law of motion of η_t given by (16).

The economy is stable near the upper boundary $\overline{\eta}$ where the level of credit provision is the highest (see Proposition 7). According to Proposition 10, when firms' pledgeability constraint binds in high-η_t states (i.e., $\eta_t > \widetilde{\eta}$), bank leverage tends to be low, which implies that η_t is insenstive to shocks (its diffusion is given by $\sigma_t^\eta = (x_t - 1)$ in (16). Therefore, the stationary distribution density, $p(\eta_t)$, has a local maximum at $\overline{\eta}$. The local maximum of density implies that the economy tends to spend a relatively large amount of time near $\overline{\eta}$. The proof is in Appendix A.

Proposition 12 (Stable Credit Booms) *There exists a constant $\eta_B > \widetilde{\eta}$, where $\widetilde{\eta}$ is defined in Proposition 10, such that in the interval $(\eta_B, \overline{\eta})$, the stationary density, $p(\eta_t)$, is maximized at $\overline{\eta}$.*

Credit crunch. Credit crunches happen when η_t is low and near $\underline{\eta}$ where bankers pay the equity issuance cost and recapitalize. In these low-η_t states, credit supply is inelastic (Proposition 5), and credit demand is weak (Proposition 6), resulting in low levels of credit provision and investment (Proposition 7). To clarify the intuitions about crisis dynamics, I first characterize how long it takes to recover and then relate the long-run probability of crises to the dynamics of bank leverage described in Proposition 10 and that of bank ROE in Proposition 11. I consider the most severe crisis where η_t has fallen to its lower boundary $\underline{\eta}$. The recovery time, $g(\eta)$, is defined as the expected time it takes for η_t to reach any value $\eta > \underline{\eta}$ from $\underline{\eta}$, i.e., $g(\eta) \equiv \mathbb{E}\left[\tau - t \,|\, \eta_t = \underline{\eta}\right]$, where $\tau = \min\{s \geq t : \eta_s \geq \eta\}$. The proof of next lemma on recovery time is in Appendix A.

Lemma 9 (Recovery Time) *The expected time to reach η from $\underline{\eta}$, $g(\eta)$, is solved by the ODE*

$$1 - g'(\eta)\,\mu^\eta(\eta) - \frac{\sigma^\eta(\eta)^2}{2} g''(\eta) = 0,$$

with the boundary conditions, $g\left(\underline{\eta}\right) = 0$ and $g'\left(\underline{\eta}\right) = 0$.

According to Proposition 11, when η_t is low, bank ROE is low, so the accumulation of bank equity through retained earnings is slow, which already suggests that the economy tends to get stuck in low-η_t states. Moreover, according to Proposition 10, bank leverage is low when η_t is low so η_t is not very sensitive to shocks (see the law of motion (16)). A low leverage limits the impact of both positive and negative shocks on η_t. However, this dampening effect is mainly relevant for positive shocks, as the impact of negative shocks is already limited by the lower bound $\underline{\eta}$, that is negative shocks cannot push η_t down below $\underline{\eta}$ where bankers optimally pay the equity issuance cost and recapitalize. Therefore, low leverage in low-η_t states mainly serves to dampen the impact of positive shocks and thus contributes to stagnant credit crunches. In fact, the next proposition shows that the lower bank leverage is, the more sluggish recovery is out of credit crunches.

Proposition 13 (Sluggish Recovery) *The expected time to reach η from $\underline{\eta}$, $g(\eta)$ in Lemma 9, in increasing and convex (quadratic) in η. Moreover, there exists a constant $\eta_R > \underline{\eta}$ such that, for any $\eta \in \left(\underline{\eta},\, \eta_R\right)$, a decrease in bank leverage at $\underline{\eta}$ leads to an increase in $g(\eta)$.*

The proof is in Appendix A. This proposition shows that at $\eta_t = \underline{\eta}$, the expected time it takes to reach $\eta > \underline{\eta}$ increases quadratically as the destination state becomes more distant. This result offers a concrete description of the stagnant nature of crises in the model. Next, I show that the recovery path is not only slow but also fragile. The economy recovers from a crisis if bankers accumulate equity through retained earnings or experience a sequence of positive shocks that increase η_t. According to Proposition 10, when η_t is small ($< \widetilde{\eta}$), bank leverage is increasing in η_t. Therefore, as η_t increases and drives up bank leverage, η_t becomes more sensitive to shocks (the shock loading is $\sigma_t^\eta = (x_t - 1)\,\sigma$ in the law of motion (16)). The increase in shock sensitivity implies that negative shocks can easily push the economy back to low-η_t states. In summary, recovery from a crisis is slow, and the path is fragile due to leverage procyclicality. The next proposition shows that the economy gets stuck in the low-η_t states. The proof is in Appendix A.

Proposition 14 (Stagnant Credit Crunch) *There exists a constant $\eta_C < \widetilde{\eta}$, where $\widetilde{\eta}$ is defined in Proposition 10, such that in the interval $\left[\underline{\eta},\ \eta_C\right)$, the stationary density, $p(\eta_t)$, is maximized at $\underline{\eta}$.*

The economy tends to spend more time in credit booms (η_t near $\overline{\eta}$) and credit crunches (η_t near $\underline{\eta}$) accordingly to Proposition 12 and 14, respectively, as there are two modes of the stationary density at the two extreme values of η_t. A key force is the hump-shaped relationship between bank leverage and η_t. A corollary of leverage procyclicality in low-η_t states and countercyclicality in high-η_t states is that the economy spends relatively less time in the intermediate-η_t states where bank leverage is high and the economy is very responsive to shocks. In other words, the transition between high- and low-η_t states can be fairly swift. These results speak to the empirical findings on the multimodality of macro and financial conditions (Adrian, Boyarchenko, and Giannone, 2021).

4 Application: Policy Intervention

Since the Global Financial Crisis (GFC), governments around the world, including both fiscal authorities and central banks, have intervened actively to sustain credit supply during major economic downturns (Gertler and Kiyotaki, 2010). During the Covid-19 pandemic, not only the GFC-era programs were reinstated, but new programs were introduced to provide credit directly to nonfinancial firms, such as the Primary and Secondary Market Corporate Credit Facilities (PMCCF and SMCCF) in the U.S. The model provides a new framework for understanding credit interventions.

To capture the design of credit intervention in practice, I consider the following policy. The government lends $l^G(\eta_t)$ to entrepreneurs with $l^G(\eta_t) = 0$ if $\eta_t > \eta^G$ and $l^G(\eta_t) = \omega_0 - \omega_1\eta_t > 0$ for $\eta_t \leq \eta^G$ where $\omega_1 > 0$—that is, the government intervenes if the private sector's capacity to supply credit, which depends on η_t, is below a threshold, and increases the scale of intervention as η_t declines further.[34] In line with the design of PMCCF and SMCCF and other programs, credit is provided by the government at the market loan rate, R_t.[35] Therefore, intervention essentially shifts outward the credit supply curve: for any loan rate R_t, aggregate credit supply is the sum of bank-supplied credit, $\eta_t + (R_t - \rho - \delta)/b(\eta_t)$, which is obtained from rearranging the supply curve given by (20), and government-supplied credit, $l^G(\eta_t)$, i.e.,

$$l_t^S = \underbrace{\eta_t + \frac{R_t - \rho - \delta}{b(\eta_t)}}_{\textit{Bank credit}} + \underbrace{l^G(\eta_t)}_{\textit{Gov. credit}}. \tag{28}$$

When η_t decreases and bank credit declines (see Proposition 5), $l^G(\eta_t)$ increases to supplement credit supply at any R_t. In the numerical solution, the parameters of $l^G(\eta_t)$, such as η^G, ω_0, and ω_1, are set in Appendix C to reflect the scale of intervention during the Covid-19 pandemic. The goal is not a comprehensive quantitative assessment of credit policy but to illustrate theoreti-

[34] The threshold, η^G, is introduced in line with Section 13.3 of the Federal Reserve Act, which permits it in "unusual and exigent circumstances" to make loans to the private sector. The statute makes clear that in normal times the Federal Reserve is not permitted to take on private credit risk.

[35] For publicly traded bonds, the government can rely on market prices, e.g., PMCCF and SMCCF in the U.S., and if a firm's debt is not publicly traded, the government can lend alongside banks and rely on banks' pricing of loans, e.g., the Main Street Lending Program (MSLP) in the U.S. during the Covid-19 pandemic.

cal mechanisms with reasonable parameter values. Finally, note that to focus on the credit-market implications, I abstract away other forms of policy distortions: government lending is funded by a lump-sum tax on households, and repayment is returned to households via a lump-sum transfer.

As previously discussed, the allocation problem in the economy is whether the households' funds can be channeled to entrepreneurs who have investment opportunities. Banks can intermediate credit supply but are financially constrained, facing the equity issuance cost. Therefore, credit intermediation by the government can potentially improve efficiency (Lucas, 2016).

Given the new credit supply curve and the credit demand curve that is still characterized by Proposition 4, the Markov equilibrium can be similarly solved as in Section 3. Moreover, I compute the social welfare at time t that is given by the present value of future consumption streams:

$$W_t = \mathbb{E}_t \left[\int_{s=t}^{\infty} e^{-\rho(s-t)} (\alpha - \lambda l_s) K_s ds \right], \tag{29}$$

where $(\alpha - \lambda l_s) K_s ds$, the aggregate consumption at time s $(\geq t)$, is the output $\alpha K_s dt$ minus goods used for creating new capital $K_s l_s \lambda ds$.[36] Appendix D shows how to compute W_t: there exists a function $\theta(\cdot)$ such that $W_t = \theta(\eta_t) K_t$, and $\theta(\eta_t)$ can be solved from a differential equation.

Figure 5 compares the laissez-faire economy and the economy under intervention. Typically, endogenous variables are plotted against the state variable, η_t. To demonstrate the impact of credit intervention across different phases of economic downturns, I use a monotone transformation of η_t as the horizontal axis in each panel, so the x-axis is $\mu_t^K = \lambda F(l_t) - \delta$, the economic (capital) growth rate, which is an increasing function of η_t.[37] In each panel, the plot starts from a growth rate of -4% and ends at 0%. Note that -4% is not the worst growth rate. In the laissez-faire economy, the worst growth rate at $\eta_t = \underline{\eta}$ is -4.7%, while the worst growth rate is -4.4% under intervention. It is already evident that intervention improves welfare by limiting the severity of credit crunch.

Panel A of Figure 5 compares welfare of the laissez-faire economy and that under intervention across different phase of economic downturns.[38] The percentage improvement from interven-

[36] As a reminder, λds fraction of firms' investment succeed over ds while the rest turn out to be storage technology.
[37] Given credit l_t, λdt fraction of investment creates new capital over dt, and δdt fraction of capital is destroyed.
[38] The laissez-faire economy is the economy with decreasing return-to-scale investment technology in Figure 4.

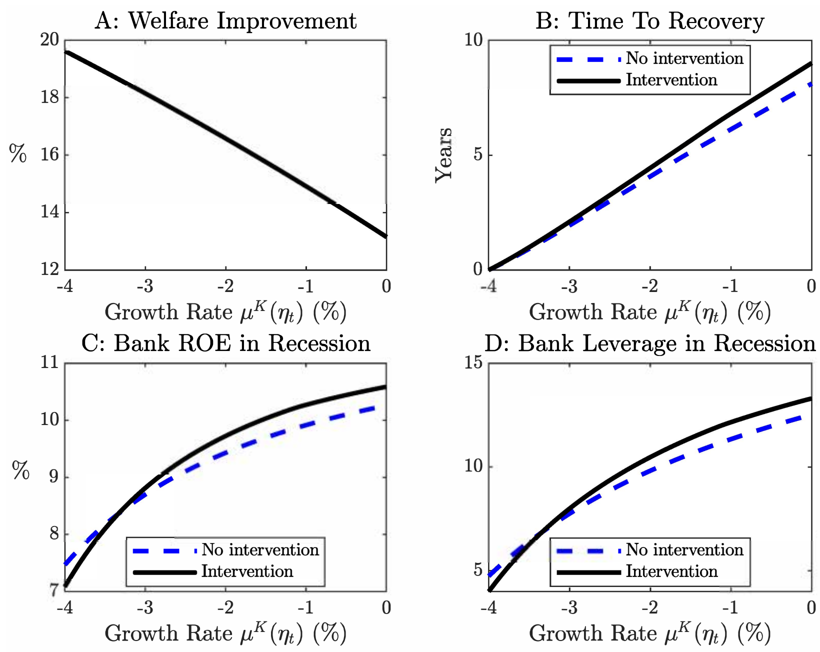

Figure 5: **Credit Intervention in Crises.** In all four panels, the horizontal axis is the expected growth rate of the economy, $\mu(\eta_t) = \lambda F(l_t) - \delta$, i.e., capital growth via the creation of new capital net off the expected capital destruction rate, which is a monotonic (positive) transformation of state variable η_t. In each panel, the plot starts from the value of η_t with $\mu(\eta_t) = -4\%$ and ends at the value of η_t with $\mu(\eta_t) = 0\%$. Panel A compares welfare of the laissez-fiare economy and that under intervention and reports the percentage improve by intervention. Panel B plots the expected time it takes to reach different levels of $\mu(\eta_t)$ starting from the value of η_t that generates $\mu(\eta_t) = -4\%$. Panel C and D report bank ROE and leverage, respectively, across different phases of a crisis (different values of $\mu(\eta_t)$).

tion is greater when growth is weaker (and η_t is lower) because, by design, the scale of intervention is larger when η_t is lower. Welfare is improved because the frictionless credit intermediation by the government partially substitutes out the frictional intermediation by banks that face the equity issuance cost. As shown in Proposition 5, the lower η_t is, the more inelastic and lower bankers' supply of credit is as costly equity issuance becomes more imminent (i.e., η_t approaches $\underline{\eta}$).

The result in Panel B of Figure 5 seems counterintuitive: government credit supply prolongs the downturn. The figure plots the expected number of years it takes to reach different levels of economic growth from the state of a severe credit crunch where the growth rate is -4%.[39] In the laissez-faire economy, it takes eight years to reach zero growth rate, while under credit intervention, it takes nine years. Section 3 explains how the model generates stagnant credit crunches that take

[39]The computation of expected time to recovery involves solving a differential equation and follows Lemma 9.

a long time to recover from, but why does government credit supply prolong credit crunches?

As long as government lending does not satiate entrepreneurs' credit demand, bank lending is needed for investment. Therefore, recovery requires the rebuild of bank equity, which in turn relies on banks' profits from lending. Intervention affects banks' lending profits through two channels. First, by shifting outward the supply curve, intervention pushes down the market-clearing loan rate along the demand curve. This force reduces banks' lending profits. Second, by making credit cheaper across the recessionary states, intervention improves the credit-financed growth of capital and thereby raises capital value. As shown in Proposition 4, a higher capital value encourages entrepreneurs to build capital and thus shift their credit demand curve outward. This force increases banks' lending profits. The first force denominates in low η_t states in Panel C of Figure 5, so bank ROE is negatively impacted by intervention, which slows down the rebuild of bank equity.

The last result is about its impact on bank leverage cyclicality and how such impact translates into not only a sluggish recovery from crises but also a fragile one. Panel D of Figure 5 shows bank leverage is more procyclical under intervention: bank leverage is lower under intervention when η_t and economic growth are near the lowest, and as η_t increases and the economy recovers, bank leverage becomes higher under intervention than the laissez-faire case. By reducing lending profits in the low-η_t states, intervention dampens bankers' incentive to maintain a high leverage. The next proposition summarizes such policy-induced deleveraging. The proof is in Appendix A.

Proposition 15 (Policy-Induced Deleveraging) *There exists a neighborhood of $\underline{\eta}$ (low-η_t states) where, as credit intervention reduces the loan rate, R_t, it also decreases bank leverage, x_t.*

In Panel D of Figure 5, bank leverage under intervention rises above the laissez-faire case as η_t increases and the economy recovers. In both economies, banks pay the issuance cost when raising equity at the lower bound of η_t. To compensate the financing costs, banks must generate sufficient profits. Under credit intervention, banks do so by raising leverage in high-η_t states where lending profits are relatively high and reducing leverage in low-η_t states where lending profits are low due to credit intervention. In other words, to sustain profits, banks shift their risk-taking capacity towards high-η_t states, resulting in more procyclical leverage under credit intervention.

The dynamics of bank leverage in Panel D of Figure 5 suggests that credit intervention not only prolongs economic downturns, as shown in Panel B, but also makes recovery more fragile. Panel B of figure 5 is about the average recovery path and does not reflect the build-up of fragility along the path. A low bank leverage at the depth of a credit crunch suggests that the state variable, η_t, is not sensitive to shocks and the economy is relatively stable in this region of low private-sector credit capacity and low growth.[40] As η_t increases and the economy recovers, bank leverage rises faster under credit intervention, and the economy becomes increasingly more sensitive to shocks.

In summary, the comparison between the laissez-faire economy and the economy under intervention shows the following results. First, intervention improves welfare. Second, intervention, by reducing banks' lending profits and slowing down the rebuild of bank equity, extends the duration of credit crunches. Third, by amplifying the procyclicality of bank leverage along the recovery path, intervention makes recovery more fragile. In the model, agents are risk-neutral. For risk-averse agents, welfare may decrease due to intervention as it prolongs the downturns and renders recovery more fragile. This paper does not analyze optimal intervention, but the mechanism it illustrates shows intervention can be harmful in more general settings where agents are risk-averse.

Admittedly, should the government satiates credit demand, providing so much credit that the marginal value of entrepreneurs' investment is zero and the first-best investment (given by Proposition 1) is achieved, the economy would no longer need bank credit, and the fact that intervention crowds out bank profits and slows down the rebuild of bank equity would be irrelevant. However, it is unlikely that the government can overtake private-sector credit intermediaries as the sole credit supplier, because in reality, the government faces constraints on its capacity to lend.

Discussion: banking regulations. Capital requirements and leverage regulations may mitigate the problems caused by credit intervention. For example, regulations that limit bank leverage reduce leverage procyclicality along the recovery path, making recovery less shock-sensitive. This force acts against the impact of credit intervention that amplifies intermediaries' leverage procyclicality coming out of crises. However, forcing banks to keep leverage low also reduces their expected return on equity and expected growth rate of net worth, which then slows down recovery.

[40]As a reminder, the diffusion or shock sensitivity in the law of motion of η_t given by (16) is $(x_t - 1)\sigma$.

5 Conclusion

In the model, crises originate on the credit-supply side: because financial intermediaries cannot be frictionlessly recapitalized, shocks have persistent impact on their net worth and lending capacity. In response, firms' demand for credit that finances capital creation weakens, as they anticipate tighter credit supply in the future that constrains capital growth. Credit-demand contraction reduces intermediaries' profitability and slows their net-worth rebuild. Intermediaries' investment opportunities deteriorate when they become undercapitalized, which distinguishes the model from those that also have intermediaries' net worth as the state variable but rely on the fire-sale dynamics.[41] Credit-demand contraction during downturns is driven not by sentiment but expectations of persistent credit-supply tightening. Crisis severity and duration are mutually reinforcing.

Two features of credit demand contribute to the emergence of two regimes (booms and crises) with transitions depending on intermediaries' leverage choice. The first feature is about how much firms *want to borrow*, which is tied to their expectations of future credit availability and capital growth. When intermediaries' net worth is low, this force creates the comovement between intermediaries' net worth and lending profits. It locks the economy in crises and contributes to the procyclicality of intermediaries' leverage, making recovery fragile. The second feature is about how much firms *can borrow*. When intermediaries' net worth is high and credit is abundant, firms hit debt limits, which in turn constrains intermediaries' asset expansion, making their leverage countercyclical. The hump-shaped leverage dynamics imply that booms and crises tend to be sticky and transitions abrupt, because intermediaries' leverage and shock sensitivity of their net worth—the state variable—are highest when intermediaries' net worth is at the intermediate levels.

[41] In the absence of asset fire sale, intermediaries' investment opportunities also improve in crises through decreasing return-to-scale: when intermediaries are undercapitalized, they lend less, so their marginal return from lending is higher (e.g., Klimenko et al. (2016)). A related insight is also highlighted by Gersbach, Rochet, and Scheffel (2023).

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Internet Appendix

A Proofs

Proof of Lemma 1 and Proposition 4 (entrepreneur optimization). Entrepreneurs maximize life-time utility, $\mathbb{E}\left[\int_{t=0}^{+\infty} e^{-\rho t} dc_t^{\mathbb{E}}\right]$, subject to the following wealth dynamics:

$$dw_t = -dc_t^{\mathbb{E}} + \mu_t^w w_t dt + \sigma_t^w w_t dZ_t + (\widehat{w}_t - w_t)\, dN_t,$$

where μ_t^w and σ_t^w are the drift and diffusion terms that will be elaborated later, dN_t is the increment of idiosyncratic Poisson process ($dN_t = 1$ if investment creates new capital), and $\widehat{w}_t$ is the post-Poisson wealth. I conjecture that the value function is linear in wealth w_t: $V_t = \zeta_t w_t$, where ζ_t is the marginal value of wealth, and in equilibrium, follows a diffusion process:

$$d\zeta_t = \zeta_t \mu_t^\zeta dt + \zeta_t \sigma_t^\zeta dZ_t,$$

where μ_t^ζ and σ_t^ζ are the drift and diffusion terms respectively. Note that ζ_t only depends on the aggregate dynamics, so it does not jump at an individual entrepreneur's Poisson shock.

Under this conjecture, the Hamilton-Jacobi-Bellman (HJB) equation is

$$\rho V_t dt = \max_{\{q_t^K k_t + dc_t^{\mathbb{E}} \le w_t,\, l_t \le q_t^K\}} dc_t^{\mathbb{E}} - \zeta_t dc_t^{\mathbb{E}} + \left\{ w_t \zeta_t \mu_t^\zeta + w_t \zeta_t \mu_t^w + w_t \zeta_t \sigma_t^\zeta \sigma_t^w \right\} dt + \zeta_t \left[\widehat{w}_t - w_t\right] dt.$$

At any t, entrepreneurs allocate wealth between consumption and savings in the form of capital ownership. Let ψ_t denote the cumulative Lagrange multiplier of the budget constraint, so at t, $d\psi_t$ is the shadow price of $q_t^K k_t + dc_t^{\mathbb{E}} \le w_t$. After deciding on capital ownership, k_t, entrepreneurs choose the amount of borrowing from the bank for each unit of capital subject to the pledgeability constraint, $l_t \le q_t^K$. Let ξ_t denote the cumulative value of Lagrange multiplier for the pledgeability constraint, so at t, $d\xi_t$ is the shadow price of $l_t \le q_t^K$.

In equilibrium, all entrepreneurs hold finite amount of capital, so they are indifferent between consumption and saving. Thus, the marginal value of savings, ζ_t, is equal to one (i.e., the marginal value of consumption), and, as a result, μ_t^ζ and σ_t^ζ are zero. The HJB equation is simplified under $\mu_t^\zeta = 0$ and $\sigma_t^\zeta = 0$, and is augmented with the constraints and multipliers:

$$\rho V_t dt = \max_{\{q_t^K k_t + dc_t^{\mathbb{E}} \le w_t, l_t \le q_t^K\}} \mu_t^w w_t dt + \lambda \left[\widehat{w}_t - w_t\right] dt + \left(w_t - q_t^K k_t\right) d\psi_t + k_t \left(q_t^K - l_t\right) d\xi_t.$$

Given k_t and l_t, the wealth drift is

$$\mu_t^w w_t dt = \alpha k_t dt + \mathbb{E}_t \left(q_{t+dt}^K k_{t+dt} - q_t^K k_t \right) - (R_t - \delta) l_t k_t dt,$$

and the wealth jump at the Poisson time is

$$\widehat{w}_t - w_t = q_t^K F(l_t) k_t - l_t k_t.$$

The first term in $\mu_t^w w_t dt$ is the production flow from capital, and the second term is the appreciation and depreciation from the stochastic destruction and potential reevaluation of capital value. The loan will be repaid if capital survives; with probability $\delta dt - \sigma dZ_t$, capital perishes and the entrepreneur defaults on the bank loan. Therefore, the cost is $[1 - (\delta dt - \sigma dZ_t)](1 + dR_t dt) - 1$, which in expectation is $(R_t - \delta)$. Second, consider the loan benefit. With probability λdt, the investment creates new capital that generates a jump in wealth.

The optimality conditions for l_t are: $d\xi_t \geq 0$ and equality holds if $l_t < q_t^K$, and

$$-(R_t - \delta) k_t dt + \left[q_t^K F'(l_t) - 1 \right] k_t \lambda dt - k_t d\xi_t = 0,$$

Rearranging the equation and dividing both sides of the equation by $k_t dt$, I obtain

$$R_t - \delta = \lambda \left[q_t^K F'(l_t) - 1 \right] - d\xi_t / dt.$$

Therefore, when the pledgeability constraint is not binding, $d\xi_t = 0$, I obtain the optimality condition (8) in Lemma 1. When the pledgeability constraint binds, $l_t = q_t^K$ and it is possible that the shadow price of pledgeability constraint is positive, i.e., $d\xi_t > 0$ and $R_t - \delta < \lambda \left[q_t^K F'(l_t) - 1 \right].$

Proof of Lemma 2. Next, I continue from the derivation in the proof of Lemma 1 and Proposition 4 and solve the asset pricing equation (11) for capital. A fraction $(\delta dt - \sigma dZ_t)$ of capital is to be destroyed, so the capital evolves as

$$k_{t+dt} = k_t - (\delta dt - \sigma dZ_t) k_t.$$

Given the equilibrium capital value dynamics, $dq_t^K = q_t^K \mu_t^K dt + q_t^K \sigma_t^K dZ_t$, I obtain

$$q_{t+dt}^K k_{t+dt} - q_t^K k_t = q_t^K k_t \left[-(\delta dt - \sigma dZ_t) + \mu_t^K dt + \sigma_t^K dZ_t + \sigma \sigma_t^K dt \right].$$

The optimality conditions for k_t are: $d\psi_t \geq 0$ and equality holds if $q_t^K k_t + dc_t^{\mathbb{E}} < w_t$, and

$$\alpha dt + q_t^K \left(-\delta + \mu_t^K + \sigma \sigma_t^K \right) dt - (R_t - \delta) l_t dt + \left[q_t^K F(l_t) - l_t \right] \lambda dt - q_t^K d\psi_t + \left(q_t^K - l_t \right) d\xi_t = 0.$$

Substituting these optimality conditions into the HJB equation, I obtain

$$\rho V_t dt = w_t d\psi_t.$$

Because $\zeta_t = 1$, I obtain $V_t = w_t$, and $d\psi_t = \rho dt$. Note that $\left(q_t^K - l_t\right) d\xi_t = 0$ from the optimality condition for l_t. Substituting $d\psi_t = \rho dt$ into the F.O.C. for k_t and rearranging the equation, I obtain

$$q_t^K = \frac{\alpha - (R_t - \delta)\, l_t + \lambda \left[q_t^K F\left(l_t\right) - l_t\right]}{\rho - \left(\mu_t^K - \delta + \sigma \sigma_t^K\right)}.$$

Proof of Proposition 1. In Equation (11), $R_t - \delta$ is replaced by ρ, the equilibrium effective loan rate (i.e., the households' discount rate). When l_{FB} is a constant, the numerator of the right side of (11) is constant and, thus, $\mu_t^K = 0$ and $\sigma_t^K = 0$ and $q_t^K = q_{FB}^K$ (constant) are consistent. When $q_t^K = q_{FB}^K$ is a constant, l_{FB} is a constant. Therefore, there exists a first-best equilibrium where l_{FB} and q_{FB}^K are constant and given by (13) and (12), respectively.

Proof of Proposition 2. First, I show that when q_t^B and q_t^K are functions of η_t, the market-clearing level of credit per unit of capital, l_t, is also a function of η_t, which then implies that the market-clearing loan rate, R_t, is a function of η_t. Moreover, l_t being a function of η_t also implies that bank leverage (loan-to-equity ratio), $l_t K_t / E_t = l_t / \eta_t$, is a function of η_t. Therefore, from the law of motion of η_t given by (16) is autonomous (note that the interest rate offered by bank debt, r_t, is equal to ρ, households' discount rate, in equilibrium). To show that l_t is a function of η_t when q_t^B and q_t^K are functions of η_t, consider two cases. If the pledgeability constraint binds, $l_t = q_t^K$ and therefore is a function of η_t. If the pledgeability constraint does not bind, the market-clearing condition given by (23) implies that l_t is a function of η_t (note that $b(\eta_t)$ is defined in (21)). In summary, I have shown that when q_t^B and q_t^K are functions of η_t, the equilibrium loan amount, l_t, loan rate, R_t, and bank leverage, x_t, are all functions of η_t, and therefore, η_t has an autonomous law of motion.

Next, I show that when q_t^B and q_t^K are functions of η_t, the equation (11) for capital valuation and bankers' HJB equation (A.1), which can be simplified to (A.4), $\mu_t^B = \overline{\rho} - \rho$, constitute a pair of ordinary differential equations (ODEs) for $q^K(\eta_t)$ and $q^B(\eta_t)$, i.e., a unique mapping from $\left(\eta, q^B, q^K, \frac{dq^B}{d\eta}, \frac{dq^K}{d\eta}\right)$ to $\left(\frac{d^2 q^B}{d\eta^2}, \frac{d^2 q^K}{d\eta^2}\right)$. Therefore, the conjecture that q_t^B and q_t^K are functions of η_t can be successful verified to be consistent with the equilibrium conditions, as q_t^B and q_t^K can be solved as functions of η_t via the ODE system.[42] Instead of first derivatives, I work with elasticities of $\left(q^B, q^K\right)$, $\epsilon^X = \frac{dq^X/q^X}{d\eta/\eta}$, $X \in \{B, K\}$ to simplify expressions. By Itô's lemma, I obtain

$$\mu^X = \epsilon^X \mu^\eta + \frac{1}{2q^X} \left(\sigma^\eta \eta\right)^2 \frac{d^2 q^X}{d\eta^2}, \text{ i.e., } \frac{d^2 q^X}{d\eta^2} = 2q^X \frac{\left(\mu^X - \epsilon^X \mu^\eta\right)}{\left(\sigma^\eta \eta\right)^2}, X \in \{B, K\}.$$

I have shown that when q_t^B and q_t^K are functions of η_t, η_t has an autonomous law of motion, that is the drift, μ^η, an diffusion, σ^η, are both functions of η_t. Thus, from (A.4), I obtain the first ODE. The equation (11) for capital valuation implies that second ODE, because, first, I have sown that when q_t^B and q_t^K are functions of η_t, the credit market-clearing price and quantity, R_t and l_t, are functions of η_t, and second, by Itô's lemma, $\sigma^K = \epsilon^K \sigma^\eta$. There are two sets of boundary conditions for this pair of ODEs. The first set contains the optimality conditions for bankers' choices of equity raising and consumption: $\epsilon^B(\underline{\eta}) = -1$, $\epsilon^B(\overline{\eta}) = 0$, $q^B(\underline{\eta}) = 1 + \chi$, $q^B(\overline{\eta}) = 1$. These conditions are discussed in further details in the proofs below that characterize bankers' optimization.[43] The second set of boundary conditions include $\epsilon^K(\underline{\eta}) = 0$ and $\epsilon^K(\overline{\eta}) = 0$ that rule out arbitrage at the reflecting boundaries $\underline{\eta}$ and $\overline{\eta}$ where the sign of $d\eta_t$ is predictable. Therefore, I obtain a total of six boundaries conditions for the two second-order ODEs and two endogenous boundaries, $\underline{\eta}$ and $\overline{\eta}$.

In summary, I have shown that q_t^B and q_t^K can be solved as functions of η_t and, given that q_t^B and q_t^K are functions of η_t, l_t, R_t, and x_t are all functions of η_t and η_t has an autonomous law of motion. Thus, a time-homogeneous Markov equilibrium exists.

Proof of Lemma 3, 4, and 7 and Proposition 3 and 5 (banker optimization). First, I derive the law of motion of η_t give by (16). Banks' equity have the same drift term, $\mu_t^{\mathbb{E}}$,

$$\mu_t^{\mathbb{E}} \equiv r_t + x_t \left(R_t - \delta - r_t\right) dt\,,$$

and the same diffusion term,

$$\sigma_t^{\mathbb{E}} \equiv x_t \sigma\,,$$

of equity and their payout/issuance rates are the same dy_t, so aggregating over banks, the law of motion of aggregate bank equity E_t is given by

$$dE_t = \mu_t^{\mathbb{E}} E_t dt + \sigma_t^{\mathbb{E}} E_t dZ_t - dy_t E_t.$$

[42] Equations (A.1) and (A.4), though appear later in the appendix, do not require that q_t^B and q_t^K are functions of η_t.
[43] These set of conditions are similar to those in Brunnermeier and Sannikov (2014) and Phelan (2016).

Given the expected growth rate, $\lambda F(l_t) - \delta$, which is the investment net of expected depreciation, the aggregate capital stock, K_t, evolves as: $dK_t = [\lambda F(l_t) - \delta] K_t dt + \sigma K_t dZ_t$. By Itô's lemma, the ratio, $\eta_t = \frac{E_t}{K_t}$, has the following law of motion:

$$d\eta_t = \frac{1}{K_t} dE_t - \frac{E_t}{K_t^2} dK_t + \frac{1}{K_t^3} \langle dK_t, dK_t \rangle - \frac{1}{K_t^2} \langle dE_t, dK_t \rangle ,$$

where $\langle dX_t, dY_t \rangle$ denotes the quadratic covariation of diffusion processes X_t and Y_t, so I obtain $\langle dK_t, dK_t \rangle = \sigma^2 K_t^2 dt$, and $\langle dE_t, dK_t \rangle = \sigma_t^{\mathbb{E}} \sigma E_t K_t dt$. Dividing both sides by η_t, I obtain

$$\frac{d\eta_t}{\eta_t} = \frac{dE_t}{E_t} - \frac{dK_t}{K_t} + \sigma^2 dt - \sigma_t^{\mathbb{E}} \sigma dt.$$

Substituting the law of motions of E_t and K_t into the equation above, I obtain Equation (16).

Next, I solve the bankers' optimization problem. Conjecture the bank's value function is linear in equity: $v_t = q_t^B e_t$. In equilibrium, the marginal value of equity, q_t^B, evolves as follows

$$dq_t^B = q_t^B \mu_t^B dt + q_t^B \sigma_t^B dZ_t.$$

Let dy_t denote dc_t^B / e_t, a consumption-to-wealth ratio. Under this conjecture, the HJB equation is

$$\begin{aligned} \overline{\rho} v_t dt &= \max_{dy_t \in \mathbb{R}} \left\{ \left(1 - q_t^B\right) \mathbb{I}_{\{dy_t > 0\}} e_t dy_t + \left(q_t^B - 1 - \chi\right) \mathbb{I}_{\{dy_t < 0\}} e_t \left(-dy_t\right) \right\} \\ &\quad + \mu_t^B q_t^B e_t dt + \max_{x_t \geq 1} \left\{ r_t + x_t \left(R_t - \delta - r_t\right) - x_t \gamma_t^B \sigma \right\} q_t^B e_t dt \, . \end{aligned}$$

where $\gamma_t^B = -\sigma_t^B$. Dividing both sides by $q_t^B e_t$, we eliminate e_t in the HJB equation,

$$\begin{aligned} \overline{\rho} &= \max_{dy_t \in \mathbb{R}} \left\{ \frac{\left(1 - q_t^B\right)}{q_t^B} \mathbb{I}_{\{dy_t > 0\}} dy_t + \frac{\left(q_t^B - 1 - \chi\right)}{q_t^B} \mathbb{I}_{\{dy_t < 0\}} \left(-dy_t\right) \right\} \\ &\quad + \mu_t^B + \max_{x_t \geq 1} \left\{ r_t + x_t \left(R_t - \delta - r_t\right) - x_t \gamma_t^B \sigma \right\} , \end{aligned} \tag{A.1}$$

which confirms the conjecture of linear value function.

When paying out dividend, the bankers receive 1 for consumption, but lose q_t^B. Only when $q_t^B \leq 1$, $dy_t > 0$. When the bank issues equity, it incurs a dilution cost. From the existing shareholders' perspective, one dollar equity is sold to outside investors at a discount price $\frac{q_t^B}{1+\chi}$. To raise $(-dy_t) e_t$ that is worth $q_t^B (-dy_t) e_t$, the bank must issue $\frac{(1+\chi)(-dy_t)e_t}{q_t^B}$ shares, and thus, the existing shareholders lose value of $q_t^B \frac{(1+\chi)(-dy_t)e_t}{q_t^B} = (1+\chi)(-dy_t) e_t$. Therefore, the bank raises equity only if $q_t^B \geq 1+\chi$. the marginal value of equity, q_t^B, is a function of the state variable η_t. Bankers' payout and issuance policies imply that $q_t^B \in [1, 1+\chi]$; otherwise, bankers will pay out or issue an infinite amount. Let $\underline{\eta}$ denote the state where $q_t^B = 1+\chi$, and $\overline{\eta}$ denote the state where $q_t^B = 1$. It must hold that $\underline{\eta} < \overline{\eta}$ and $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right]$ because payout moves η_t at $\underline{\eta}$ upward and issuance moves η_t downward at $\overline{\eta}$. The amounts of issuance at $\underline{\eta}$ and payout at $\overline{\eta}$ exactly offset any variation in η_t that would otherwise cause η_t to move beyond the two reflecting boundaries $\underline{\eta}$ and $\overline{\eta}$.

The optimality of dy_t also require that bank owners' value cannot be further improved at the boundaries.[44] The value of bank equity is $q_t^B E_t = q_t^B \eta_t K_t$ where K_t is not affected by dy_t. At the issuance boundary $\underline{\eta}$, $\frac{d\left(q_t^B \eta_t\right)}{d\eta_t} = 0$, which is equivalent to

$$\epsilon_t^B = -1\,, \tag{A.2}$$

guarantees the marginal impact of issuance on the value of existing shares is zero. At the payout boundary $\overline{\eta}$, $\frac{d\left(q_t^B \eta_t\right)}{d\eta_t} = 1$, which is equivalent to

$$\epsilon_t^B = 0\,, \tag{A.3}$$

or $\frac{dq^B(\eta_t)}{d\eta_t} = 0$, guarantees the value of equity declines exactly by the amount of dividends paid out. If these conditions are violated, bankers would adjust the timing of issuance and payout given the reflection of η_t, inconsistent with $\underline{\eta}$ and $\overline{\eta}$ being the issuance and payout boundaries, respectively.

From the HJB equation, the optimality condition for bank leverage x_t is:

$$R_t - \delta - r_t - \gamma_t^B \sigma \ \leq 0,$$

which is Equation (18).[45] The result $b_t = 0$ follows from $\frac{dq^B(\eta_t)}{d\eta_t} = 0$ at $\overline{\eta}$.

Substituting the optimality conditions for payout/issuance, dy_t, and leverage, x_t, and $r_t = \rho$, the households' optimality condition for holding deposits, into the HJB equation, I obtain

$$\mu_t^B = \overline{\rho} - \rho \equiv \iota. \tag{A.4}$$

Next, I prove $\epsilon_t^B \in [-1, 0]$, $\frac{d\epsilon_t^B}{d\eta_t} > 0$, and $\frac{d\left(-\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t} < 0$ (i.e., $\frac{db_t}{d\eta_t} < 0$ in Proposition 5). $\epsilon_t^B \in [-1, 0]$ and ϵ_t^B monotonically increases in η_t imply: (1) at any $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right)$, $\epsilon_t^B < 0$ and q_t^B is

[44] See also Brunnermeier and Sannikov (2014) and Phelan (2016) for similar conditions.

[45] Note that $x_t > 1$, i.e., bankers borrow, because bankers are more impatient than households ($\overline{\rho} > \rho$). Therefore, the equality always holds in equilibrium.

decreasing in η_t (in Lemma 3); (2) $b_t > 0$ in $\left[\underline{\eta}, \overline{\eta}\right)$ and $b_t = 0$ at $\overline{\eta}$ (in Proposition 3).

By definition, $\epsilon_t^B = \frac{dq_t^B}{d\eta_t}\frac{\eta_t}{q_t^B}$, so I obtain

$$\eta_t \frac{d\epsilon_t^B}{d\eta_t} = \frac{d^2 q_t^B}{d\eta_t^2}\frac{\eta_t^2}{q_t^B} - \left(\frac{dq_t^B}{d\eta_t}\frac{\eta_t}{q_t^B}\right)^2 + \frac{dq_t^B}{d\eta_t}\frac{\eta_t}{q_t^B}.$$

Then using Itô's lemma on μ_t^B, I obtain $\frac{d^2 q_t^B}{d\eta_t^2}\frac{\eta_t^2}{q_t^B} = \frac{2}{\left(\sigma_t^\eta\right)^2}\left(\mu_t^B - \epsilon_t^B \mu_t^\eta\right)$. Substituting out $\frac{d^2 q_t^B}{d\eta_t^2}\frac{\eta_t^2}{q_t^B}$ using this expression and $\frac{dq_t^B}{d\eta_t}\frac{\eta_t}{q_t^B}$ by ϵ_t^B, I obtain

$$\eta_t \frac{d\epsilon_t^B}{d\eta_t} = \frac{2}{\left(\sigma_t^\eta\right)^2}\left(\mu_t^B - \epsilon_t^B \mu_t^\eta\right) - \left(\epsilon_t^B\right)^2 + \epsilon_t^B. \tag{A.5}$$

Given the optimality condition for bankers' choice of x_t, i.e., $R_t - \delta - r_t = \gamma_t^B \sigma$ and the equilibrium deposit rate $r_t = \rho$, μ_t^η can be written as

$$\mu_t^\eta = \rho + x_t \gamma_t^B \sigma - \left[\lambda F\left(l_t\right) - \delta\right] - \sigma_t^\eta \sigma.$$

Note that $\sigma_t^\eta = x_t \sigma - \sigma$, so

$$\mu_t^\eta = \rho - \left[\lambda F\left(l_t\right) - \delta\right] + \gamma_t^B \sigma_t^\eta + \gamma_t^B \sigma - \sigma_t^\eta \sigma.$$

Substituting $\gamma_t^B = -\epsilon_t^B \sigma_t^\eta$ (from Itô's lemma) into the expression above, I obtain

$$\mu_t^\eta = \rho - \left[\lambda F\left(l_t\right) - \delta\right] - \epsilon_t^B \left(\sigma_t^\eta\right)^2 - \left(\epsilon_t^B + 1\right)\sigma_t^\eta \sigma. \tag{A.6}$$

On the right side of Equation (A.5) substituting out μ_t^η using (A.6), I obtain

$$\begin{aligned}
\eta_t \frac{d\epsilon_t^B}{d\eta_t} =& \frac{2}{\left(\sigma_t^\eta\right)^2}\left(\mu_t^B - \epsilon_t^B \mu_t^\eta\right) - \left(\epsilon_t^B\right)^2 + \epsilon_t^B \\
=& \frac{2\mu_t^B - 2\epsilon_t^B \left\{\rho - \left[\lambda F\left(l_t\right) - \delta\right]\right\}}{\left(\sigma_t^\eta\right)^2} + 2\left(\epsilon_t^B\right)^2 + \frac{2\epsilon_t^B\left(\epsilon_t^B + 1\right)\sigma}{\sigma_t^\eta} - \left(\epsilon_t^B\right)^2 + \epsilon_t^B.
\end{aligned}$$

Note that $\rho - \left[\lambda F\left(l_t\right) - \delta\right] > \frac{\sigma^2}{2}$ due to the parameter restriction $\rho > \lambda \max\left\{F\left(l\right)\right\} - \delta + \frac{\sigma^2}{2}$. Therefore, substituting $\rho - \left[\lambda F\left(l_t\right) - \delta\right]$ by $\frac{\sigma^2}{2}$ on the right side and simplifying it, I obtain

$$\eta_t \frac{d\epsilon_t^B}{d\eta_t} = \frac{2}{\left(\sigma_t^\eta\right)^2}\left(\mu_t^B - \epsilon_t^B \mu_t^\eta\right) - \left(\epsilon_t^B\right)^2 + \epsilon_t^B \geq \frac{2\mu_t^B - \epsilon_t^B \sigma^2}{\left(\sigma_t^\eta\right)^2} + \left(\epsilon_t^B\right)^2 + \frac{2\epsilon_t^B\left(\epsilon_t^B + 1\right)\sigma}{\sigma_t^\eta} + \epsilon_t^B.$$

With $\sigma_t^\eta = x_t\sigma - \sigma$, the inequality can be further simplified to

$$\eta_t \frac{d\epsilon_t^B}{d\eta_t} \geq \frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2} - \frac{\sigma^2}{\left(\sigma_t^\eta\right)^2}\left[-\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1)^2 - 2\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1) + \epsilon_t^B\right]. \quad \text{(A.7)}$$

A sufficient condition for $\frac{d\epsilon_t^B}{d\eta_t} > 0$ is that on the right side, the expression in square bracket,

$$-\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1)^2 - 2\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1) + \epsilon_t^B,$$

is non-positive. The expression is a quadratic form in $(x_t - 1)$. As a quadratic form of $(x_t - 1)$, it achieves the maximum at $x_t - 1 = -1$, i.e., $x_t = 0$, and for $x_t > 0$, this quadratic form decreases in x_t. Its value at $x_t = 0$ is $\left(\epsilon_t^B\right)^2 + 2\epsilon_t^B$. Since $x_t > 0$, I obtain

$$\left(\epsilon_t^B\right)^2 + 2\epsilon_t^B > -\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1)^2 - 2\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1) + \epsilon_t^B,$$

To show the quadratic form non-positive, it is sufficient to show its upper bound $\left(\epsilon_t^B\right)^2 + 2\epsilon_t^B \leq 0$, which requires $\epsilon_t^B \in [-2, 0]$. If $\epsilon_t^B \in [-2, 0]$, $-\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1)^2 - 2\left(\left(\epsilon_t^B\right)^2 + \epsilon_t^B\right)(x_t-1) + \epsilon_t^B \leq \left(\epsilon_t^B\right)^2 + 2\epsilon_t^B \leq 0$, so Equation (A.7) implies that $\eta_t\frac{d\epsilon_t^B}{d\eta_t} \geq \frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}$, and from Equation (A.4), $\mu^B = \iota$, so $\eta_t\frac{d\epsilon_t^B}{d\eta_t} \geq \frac{2\iota}{\left(\sigma_t^\eta\right)^2} > 0$, i.e., $\frac{d\epsilon_t^B}{d\eta_t} > 0$. Therefore, to prove $\frac{d\epsilon_t^B}{d\eta_t} > 0$, the key is to prove $\epsilon_t^B \in [-2, 0]$. Next, I will prove $\frac{d\left(-\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t} < 0$ in Proposition 5, and in the process, prove that $\epsilon_t^B \in [-1, 0] \subset [-2, 0]$ and thus $\frac{d\epsilon_t^B}{d\eta_t} > 0$.

By definition, $\frac{\epsilon_t^B}{\eta_t} = \frac{dq_t^B}{d\eta_t}\frac{1}{q_t^B}$, so

$$\frac{d\left(-\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t} = \frac{d\left(\frac{dq_t^B}{d\eta_t}\frac{1}{q_t^B}\right)}{d\eta_t} = \frac{d^2q_t^B}{d\eta_t^2}\frac{1}{q_t^B} - \left(\frac{dq_t^B}{d\eta_t}\frac{1}{q_t^B}\right)^2 = \frac{d^2q_t^B}{d\eta_t^2}\frac{1}{q_t^B} - \left(\frac{\epsilon_t^B}{\eta_t}\right)^2. \quad \text{(A.8)}$$

Similarly as before, using Itô's lemma, I obtain $\frac{d^2q_t^B}{d\eta_t^2}\frac{1}{q_t^B} = \frac{2}{\left(\sigma_t^\eta\right)^2\eta_t^2}\left(\mu_t^B - \epsilon_t^B\mu_t^\eta\right)$. Substituting out $\frac{d^2q_t^B}{d\eta_t^2}\frac{1}{q_t^B}$ with this expression and multiplying both sides of (A.8) by η_t^2, I obtain

$$\frac{d\left(\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t}\eta_t^2 = \frac{d\left(\frac{dq_t^B}{d\eta_t}\frac{1}{q_t^B}\right)}{d\eta_t}\eta_t^2 = \frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2} - \frac{2\epsilon_t^B}{\left(\sigma_t^\eta\right)^2}\mu_t^\eta - \left(\epsilon_t^B\right)^2 \quad \text{(A.9)}$$

Therefore, to prove $\frac{d\left(\frac{dq_t^B}{d\eta_t}\frac{1}{q_t^B}\right)}{d\eta_t} > 0$, it is equivalent to prove the right side is positive.

On the right side of (A.9) substituting out μ_t^η using Equation (A.6), I obtain

$$\frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}-\frac{2\epsilon_t^B}{\left(\sigma_t^\eta\right)^2}\mu_t^\eta-\left(\epsilon_t^B\right)^2 \geq \frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}-\frac{2\epsilon_t^B}{\left(\sigma_t^\eta\right)^2}\left\{\rho-\left[\lambda F\left(l_t\right)-\delta\right]\right\}+2\left(\epsilon_t^B\right)^2+\frac{2\epsilon_t^B\left(\epsilon_t^B+1\right)\sigma}{\sigma_t^\eta}-\left(\epsilon_t^B\right)^2. \tag{A.10}$$

Note that $\rho-\left[\lambda F\left(l_t\right)-\delta\right] > \frac{\sigma^2}{2}$ due to the parameter restriction $\rho > \lambda \max\left\{F\left(l\right)\right\}-\delta+\frac{\sigma^2}{2}$. Therefore, substituting $\rho-\left[\lambda F\left(l_t\right)-\delta\right]$ by $\frac{\sigma^2}{2}$ on the right side and simplifying it, I obtain

$$\frac{d\left(\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t}\eta_t^2=\frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}-\frac{2\epsilon_t^B}{\left(\sigma_t^\eta\right)^2}\mu_t^\eta-\left(\epsilon_t^B\right)^2 \geq \frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}-\frac{\epsilon_t^B\sigma^2}{\left(\sigma_t^\eta\right)^2}+\left(\epsilon_t^B\right)^2+\frac{2\epsilon_t^B\left(\epsilon_t^B+1\right)\sigma}{\sigma_t^\eta}. \tag{A.11}$$

With $\sigma_t^\eta = x_t\sigma-\sigma$, the right side can be simplified, so I obtain

$$\frac{d\left(\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t}\eta_t^2 \geq \frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}-\frac{\sigma^2}{\left(\sigma_t^\eta\right)^2}\left[-\left(\epsilon_t^B\right)^2x_t^2+\left(\epsilon_t^B\right)^2-2\epsilon_t^Bx_t+3\epsilon_t^B\right].$$

To prove that the right side is positive, I analyze the quadratic form $-\left(\epsilon_t^B\right)^2x_t^2+\left(\epsilon_t^B\right)^2-2\epsilon_t^Bx_t+3\epsilon_t^B$. This quadratic function of x_t achieves its maximum at $x_t=\frac{2\epsilon_t^B}{-2\left(\epsilon_t^B\right)^2}=-\frac{1}{\epsilon_t^B}$. The maximum is $\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1$, which increases in ϵ_t^B for $\epsilon_t^B > -\frac{3}{2}$. Given $\mu_t^B=\iota$ (Equation (A.4)) and the parameter restriction $\iota > \frac{\sigma^2}{2}$, I obtain $\frac{d\left(\epsilon_t^B/\eta_t\right)}{d\eta_t} > 0$ as long as $\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1$ is less or equal to one. At $\underline{\eta}$, $\epsilon_t^B=-1$ and $\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1=-1$, so the quadratic form is negative and I obtain $\frac{d\left(\epsilon_t^B/\eta_t\right)}{d\eta_t} > 0$ at $\underline{\eta}$. As η_t increases from $\underline{\eta}$, the quadratic form is less or equal to one and $\frac{d\left(\epsilon_t^B/\eta_t\right)}{d\eta_t} > 0$ as long as $\frac{\epsilon_t^B}{\eta_t}$ stays non-positive (i.e, $\epsilon_t^B \leq 0$). Next, I will prove $\epsilon_t^B \leq 0$ for any η_t by contradiction, and thereby, conclude that as η_t increases from $\underline{\eta}$, $\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1 \leq 1$.

Let $\widetilde{\eta}$ denote the lowest value of η_t at which $\frac{\epsilon_t^B}{\eta_t} > 0$ and $\epsilon_t^B > 0$. Since $\epsilon_t^B=0$ and $\frac{\epsilon_t^B}{\eta_t}=0$ at $\overline{\eta}$, so there must exist a state $\hat{\eta}_t \in \left(\tilde{\eta}_t, \overline{\eta}\right)$ where $\frac{\epsilon_t^B}{\eta_t} > 0$ and $\frac{d\left(\epsilon_t^B/\eta_t\right)}{d\eta_t} < 0$ so that $\frac{\epsilon_t^B}{\eta_t}$ can decrease from positive to zero eventually. Equation (A.11) and (A.4) ($\mu_t^B=\iota$) imply that at this state $\hat{\eta}_t$,

$$\frac{2\iota}{\left(\sigma_t^\eta\right)^2}-\frac{\sigma^2}{\left(\sigma_t^\eta\right)^2}\left[-\left(\epsilon_t^B\right)^2x_t^2+\left(\epsilon_t^B\right)^2-2\epsilon_t^Bx_t+3\epsilon_t^B\right] < 0,$$

which in turn implies that

$$\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1 \geq -\left(\epsilon_t^B\right)^2x_t^2+\left(\epsilon_t^B\right)^2-2\epsilon_t^Bx_t+3\epsilon_t^B > \frac{2\iota}{\sigma^2}.$$

This inequality implies that
$$\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1-\frac{2\iota}{\sigma^2}>0.$$

The left side is positive if and only if $\epsilon_t^B<\frac{-3-\sqrt{5+4\frac{2\iota}{\sigma^2}}}{2}$ or $\epsilon_t^B>\frac{-3+\sqrt{5+4\frac{2\iota}{\sigma^2}}}{2}$. Since $\epsilon_t^B>0$ at $\hat{\eta}_t$, the former case is relevant. Note that because of the parameter restriction $\frac{2\iota}{\sigma^2}>1$, $\frac{-3+\sqrt{5+4\frac{2\iota}{\sigma^2}}}{2}>0$ is a positive constant. Therefore, at states where $\frac{d\left(\epsilon_t^B/\eta_t\right)}{d\eta_t}<0$, I obtain $\epsilon_t^B>\frac{-3+\sqrt{5+4\frac{2\iota}{\sigma^2}}}{2}>0$. Once $\frac{\epsilon_t^B}{\eta_t}$ increases above zero, ϵ_t^B is bounded below by $\frac{-3+\sqrt{5+4\frac{2\iota}{\sigma^2}}}{2}$ and thus can never fall to zero at $\underline{\eta}$, which violates the boundary condition at $\underline{\eta}$. Therefore, there cannot exist a state $\widetilde{\eta}$ where $\frac{\epsilon_t^B}{\eta_t}>0$, so $\epsilon_t^B\le 0$ – as ϵ_t^B increases from -1 at $\underline{\eta}$, it never reaches above zero.

Given that $\epsilon_t^B\in[-1,0]$, I obtain
$$-\left(\epsilon_t^B\right)^2x_t^2+\left(\epsilon_t^B\right)^2-2\epsilon_t^Bx_t+3\epsilon_t^B\le\left(\epsilon_t^B\right)^2+3\epsilon_t^B+1\le 1,\quad\text{and}$$
$$\frac{d\left(\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t}\eta_t^2\ge\frac{2\mu_t^B}{\left(\sigma_t^\eta\right)^2}-\frac{\sigma^2}{\left(\sigma_t^\eta\right)^2}\left[-\left(\epsilon_t^B\right)^2x_t^2+\left(\epsilon_t^B\right)^2-2\epsilon_t^Bx_t+3\epsilon_t^B\right]\ge\frac{2\iota-\sigma^2}{\left(\sigma_t^\eta\right)^2}>0.\tag{A.12}$$

I have proven $\epsilon_t^B\in[-1,0]$, which implies $\frac{d\epsilon_t^B}{d\eta_t}>0$ as previously analyzed, and $\frac{d\left(\frac{\epsilon_t^B}{\eta_t}\right)}{d\eta_t}>0$.

Proof of Lemma 5. Please refer to the main text.

Proof of Proposition 6. At $\underline{\eta}$ and $\overline{\eta}$, $\frac{dq^K(\eta_t)}{d\eta_t}=0$ is from the boundary condition required to rule out arbitrage opportunities. $\underline{\eta}$ and $\overline{\eta}$ are reflecting boundaries, so the variation of η_t is one-sided and $\frac{dq^K(\eta_t)}{d\eta_t}\neq 0$ implies certain profits.

Next, I prove $\frac{dq_t^K}{d\eta_t}>0$ for $\eta_t\in\left(\underline{\eta},\overline{\eta}\right)$. Rearranging the optimality condition (11) for k_t, I obtain
$$q_t^K\left(\rho+\delta-\mu_t^K-\sigma_t^K\sigma\right)=\alpha+\lambda\left[q_t^KF\left(l_t\right)-l_t\right]-\left(R_t-\delta\right)l_t.\tag{A.13}$$

Note that the optimality condition for l_t in Lemma 1 implies that, when the pledgeability constraint is binding, the derivative of $\lambda\left[q_t^KF\left(l_t\right)-l_t\right]-\left(R_t-\delta\right)l_t$ with respect to l_t is non-negative, denoted by ξ_t (i.e., the shadow price of pledgeability constraint), and, when the pledgeability constraint is not binding, the derivative of $\lambda\left[q_t^KF\left(l_t\right)-l_t\right]-\left(R_t-\delta\right)l_t$ with respect to l_t is zero (i.e., $\xi_t=0$). Differentiating both sides with respect to η_t, I obtain
$$\frac{dq_t^K}{d\eta_t}\left(\rho-\lambda F\left(l_t\right)+\delta\right)-\frac{dq_t^K}{d\eta_t}\mu_t^K-\frac{dq_t^K}{d\eta_t}\sigma_t^K\sigma-q_t^K\frac{d\mu_t^K}{d\eta_t}-q_t^K\sigma\frac{d\sigma_t^K}{d\eta_t}=-l_t\frac{dR_t}{d\eta_t}+\xi_t.\tag{A.14}$$

Next, I examine the terms on the left side. First, given that

$$\mu_t^K = \frac{1}{2}\frac{d^2 q_t^K}{d\eta_t^2}\frac{1}{q_t^K}\left(\sigma_t^\eta \eta_t\right)^2 + \frac{dq_t^K}{d\eta_t}\frac{1}{q_t^K}\left(\mu_t^\eta \eta_t\right),$$

I obtain

$$\begin{aligned}\frac{d\mu_t^K}{d\eta_t} =& \frac{1}{2}\frac{d^3 q_t^K}{d\eta_t^3}\frac{1}{q_t^K}\left(\sigma_t^\eta \eta_t\right)^2 - \frac{1}{2}\frac{d^2 q_t^K}{d\eta_t^2}\frac{dq_t^K}{d\eta_t}\frac{1}{\left(q_t^K\right)^2}\left(\sigma_t^\eta \eta_t\right)^2 + \frac{1}{2}\frac{d^2 q_t^K}{d\eta_t^2}\frac{1}{q_t^K}\frac{d\left(\sigma_t^\eta \eta_t\right)^2}{d\eta_t} \\ &+ \frac{d^2 q_t^K}{d\eta_t^2}\frac{1}{q_t^K}\left(\mu_t^\eta \eta_t\right) - \left(\frac{dq_t^K}{d\eta_t}\right)^2 \frac{1}{\left(q_t^K\right)^2}\left(\mu_t^\eta \eta_t\right) + \frac{dq_t^K}{d\eta_t}\frac{1}{q_t^K}\frac{d\left(\mu_t^\eta \eta_t\right)}{d\eta_t}.\end{aligned}$$

Then, from $\sigma_t^K = \frac{dq_t^K}{d\eta_t}\frac{1}{q_t^K}\left(\sigma_t^\eta \eta_t\right)$, I obtain

$$\frac{d\sigma_t^K}{d\eta_t} = \frac{d^2 q_t^K}{d\eta_t^2}\frac{1}{q_t^K}\left(\sigma_t^\eta \eta_t\right) - \left(\frac{dq_t^K}{d\eta_t}\right)^2 \frac{1}{\left(q_t^K\right)^2}\left(\sigma_t^\eta \eta_t\right) + \frac{dq_t^K}{d\eta_t}\frac{1}{q_t^K}\frac{d\left(\sigma_t^\eta \eta_t\right)}{d\eta_t}.$$

Substituting these expressions into (A.14) and simplifying it, I obtain

$$\begin{aligned}&\frac{dq_t^K}{d\eta_t}\left(\rho - \lambda F\left(l_t\right) + \delta\right) - \frac{1}{2}\frac{d^3 q_t^K}{d\eta_t^3}\left(\sigma_t^\eta \eta_t\right)^2 - \frac{1}{2}\frac{d^2 q_t^K}{d\eta_t^2}\frac{d\left(\sigma_t^\eta \eta_t\right)^2}{d\eta_t} - \frac{d^2 q_t^K}{d\eta_t^2}\left(\mu_t^\eta \eta_t\right) - \frac{dq_t^K}{d\eta_t}\frac{d\left(\mu_t^\eta \eta_t\right)}{d\eta_t} \\ &- \frac{d^2 q_t^K}{d\eta_t^2}\left(\sigma_t^\eta \eta_t\right)\sigma - \frac{dq_t^K}{d\eta_t}\frac{d\left(\sigma_t^\eta \eta_t\right)}{d\eta_t}\sigma = -l_t\frac{dR_t}{d\eta_t} + \xi_t\,.\end{aligned}$$

To simplify the notation, define $z_t \equiv \frac{dq_t^K}{d\eta_t}$, so $z_t' \equiv \frac{d^2 q_t^K}{d\eta_t^2}$ and $z_t'' \equiv \frac{d^3 q_t^K}{d\eta_t^3}$. Moreover, define $u_t = \left(\rho - \lambda F\left(l_t\right) + \delta\right) - \frac{d\left(\mu_t^\eta \eta_t\right)}{d\eta_t} - \frac{d\left(\sigma_t^\eta \eta_t\right)}{d\eta_t}\sigma$, $\hat{\sigma}_t = \sigma_t^\eta \eta_t$, and $\hat{\mu}_t = \mu_t^\eta \eta_t + \left(\sigma_t^\eta \eta_t\right)\sigma + \frac{1}{2}\frac{d\left(\sigma_t^\eta \eta_t\right)^2}{d\eta_t}$. Substituting this new notations into the equation above, I obtain

$$\frac{1}{2}\hat{\sigma}_t^2 z_t'' + \hat{\mu}_t z_t' - u_t z_t + \left(-l_t\frac{dR_t}{d\eta_t} + \xi_t\right) = 0. \tag{A.15}$$

Given $s \geq t$, define

$$\hat{z}_s = e^{-\int_{\tau=t}^{s} u_\tau d\tau} z_s + \int_{j=t}^{s} e^{-\int_{\tau=t}^{j} u_\tau d\tau}\left(-l_j\frac{dR_j}{d\eta_j} + \xi_t\right) dj.$$

Equation (A.15) implies that $\hat{z}_s$ is a *martingale*. Let T denote the first time after t when η_t hits either the upper or lower reflecting boundary, i.e., $T = \min\left\{s \geq t : \eta_s \in \left\{\underline{\eta}, \overline{\eta}\right\}\right\}$. Therefore,

$$z_t = \hat{z}_t = \mathbb{E}_t\left[\hat{z}_T\right],$$

so, because the boundary conditions imply that $z_T = 0$,

$$\frac{dq_t^K}{d\eta_t} = z_t = \mathbb{E}_t \left[e^{-\int_{\tau=t}^{T} u_\tau d\tau} z_T + \int_{j=t}^{T} e^{-\int_{\tau=t}^{j} u_\tau d\tau} \left(-l_j \frac{dR_j}{d\eta_j} + \xi_t \right) dj \right]$$
$$= \mathbb{E}_t \left[\int_{j=t}^{T} e^{-\int_{\tau=t}^{j} u_\tau d\tau} \left(-l_j \frac{dR_j}{d\eta_j} + \xi_t \right) dj \right].^{46} \tag{A.16}$$

Next, I use this expression of $\frac{dq_t^K}{d\eta_t}$ to prove $\frac{dq_t^K}{d\eta_t} > 0$ in $\left(\underline{\eta}, \overline{\eta}\right)$ by contradiction.

The investment function, $F(\cdot)$, is a concave function. In the following, I consider a strictly concave $F(\cdot)$ and then a linear $F(\cdot)$. Differentiating the credit demand equation in Lemma 1, I obtain

$$\frac{dR_t}{d\eta_t} = \lambda \frac{dq_t^K}{d\eta_t} F'(l_t) + \lambda q_t^K F''(l_t) \frac{dl_t}{d\eta_t}. \tag{A.17}$$

Differentiating this credit supply equation (20) and rearranging the equation, I obtain

$$\frac{dl_t}{d\eta_t} = \frac{1}{\sigma^2 b_t} \frac{dR_t}{d\eta_t} - \frac{db_t}{d\eta_t} \frac{1}{b_t} (l_t - \eta_t) + 1.$$

Using this expression to substitute out $\frac{dl_t}{d\eta_t}$ in Equation (A.17), I obtain

$$\frac{dR_t}{d\eta_t} = \frac{\lambda \frac{dq_t^K}{d\eta_t} F'(l_t) + \lambda q_t^K F''(l_t) + \lambda q_t^K F''(l_t) \left(-\frac{db_t}{d\eta_t} \frac{1}{b_t} \right) (l_t - \eta_t)}{1 - \lambda q_t^K F''(l_t) \frac{1}{\sigma^2 b_t}}. \tag{A.18}$$

Because $F'' < 0$ and $b_t > 0$ for $\eta_t \in \left(\underline{\eta}, \overline{\eta}\right)$ (Proposition 3), the denominator is positive. Because $F'' < 0$, the second term in the numerator is negative. Because $\frac{db_t}{d\eta_t} < 0$ (Proposition 5) and $l_t - \eta_t = (x_t - 1)\eta_t > 0$, the last term in the numerator is negative. Suppose $\frac{dq_t^K}{d\eta_t} \leq 0$. Then the first term in the numerator is negative, so $\frac{dR_t}{\eta_t} < 0$, which, according to (A.16), implies $\frac{dR_t}{\eta_t} > 0$. This is a contradiction. Hence, I have proved that $\frac{dq_t^K}{d\eta_t} > 0$ in $\left(\underline{\eta}, \overline{\eta}\right)$ under $F'' < 0$.

Next, consider a linear $F(\cdot)$ (i.e., $F(\cdot)$ in Section 3.4 with $F'(l_t) = \kappa$). From the optimality condition on l_t,

$$\frac{dR_t}{d\eta_t} = \lambda \frac{dq_t^K}{d\eta_t} \kappa. \tag{A.19}$$

Suppose $\frac{dq_t^K}{d\eta_t} \leq 0$ and I will show contradiction. Then $\frac{dR_t}{d\eta_t} \leq 0$ and, from (A.16), $\frac{dq_t^K}{d\eta_t} \geq 0$. Therefore, $\frac{dq_t^K}{d\eta_t} = 0$ and q_t^K is a constant. Using (8) to substitute out $R_t - \delta$ in (A.13), I obtain

$$q^K (\rho + \delta) = \alpha + \lambda \left[q^K F(l_t) - l_t \right] - \lambda \left[q^K \kappa - 1 \right] l_t, \tag{A.20}$$

where, because q^K is a constant, μ_t^K and σ_t^K are substituted by zeros. Therefore, l_t is constant and, thus, (8) implies that R_t is constant. With both l_t and R_t being constant, (20) implies that b_t increases in η_t, which contradicts Proposition 5. Therefore, $\frac{dq_t^K}{d\eta_t} > 0$.

Proof of Proposition 7. The investment function $F(\cdot)$ is concave. I will first consider the case of strictly concave $F(\cdot)$ and then the case of linear $F(\cdot)$. First, I prove the results under $F''(\cdot) < 0$. Differentiating the credit demand equation in Lemma 1,

$$\frac{dR_t}{d\eta_t} = \lambda \frac{dq_t^K}{d\eta_t} F'(l_t) + \lambda q_t^K F''(l_t) \frac{dl_t}{d\eta_t}. \tag{A.21}$$

Differentiating Equation (20), I obtain

$$\frac{dl_t}{d\eta_t} = \frac{1}{\sigma^2 b_t} \frac{dR_t}{d\eta_t} - \frac{db_t}{d\eta_t} \frac{1}{b_t} (l_t - \eta_t) + 1.$$

Using this expression to substitute out $\frac{dR_t}{d\eta_t}$ in Equation (A.21), I obtain

$$\frac{dl_t}{d\eta_t} = \frac{1}{\sigma^2 b_t} \left[\lambda \frac{dq_t^K}{d\eta_t} F'(l_t) + \lambda q_t^K F''(l_t) \frac{dl_t}{d\eta_t} \right] - \frac{db_t}{d\eta_t} \frac{1}{b_t} (l_t - \eta_t) + 1.$$

Rearranging this equation, I obtain

$$\frac{dl_t}{d\eta_t} = \frac{\frac{\lambda}{\sigma^2 b_t} \frac{dq_t^K}{d\eta_t} F'(l_t) - \frac{db_t}{d\eta_t} \frac{1}{b_t} (l_t - \eta_t) + 1}{1 - \frac{\lambda}{\sigma^2 b_t} q_t^K F''(l_t)} > 0\,. \tag{A.22}$$

At any $\eta_t < \overline{\eta}$ because in the denominator, $F''(l_t) < 0$ and $b_t > 0$ at $\eta_t < \overline{\eta}$ (Proposition 3), and in numerator, $\frac{dq_t^K}{d\eta_t} \geq 0$ (Proposition 6), $F'(l_t) > 0$, and $\frac{db_t}{d\eta_t} < 0$ at $\eta_t < \overline{\eta}$. At $\overline{\eta}$, $\lambda \left[q_t^K F'(l_t) - 1 \right] = \rho$, so l_t increases in η_t because q_t^K increases in η_t and $F(\cdot)$ is concave.

Next, I prove the results under $F''(\cdot) = 0$ (i.e., for linear $F(\cdot)$). From (8) and $\frac{dq_t^K}{d\eta_t} \geq 0$ (Proposition 6), I obtain $\frac{dR_t}{d\eta_t} \geq 0$. In (20), the left side increases in η_t, while, on the right side, b_t decreases in η_t, so it must be that l_t increases in η_t.

Lemma A.1 *Given the equation*

$$y = \frac{\alpha + H(y)}{x}, \tag{A.23}$$

$\frac{dy}{dx} < 0$ *for* $x > 0$ *if* $H(y)$ *is an increasing function and* $H'(y) < x$.

The proof of Lemma A.1 is as follows. Taking derivative with respect to x on both sides of the equation and rearranging the equation, I obtain that, if $H'(y) < x$,

$$\frac{dy}{dx} = -\frac{y}{x - H'(y)} < 0\,. \tag{A.24}$$

Proof of Proposition 8. Because $\frac{dq_t^K}{d\eta_t} > 0$ (Proposition 6), proving that $q_t^K < q_{FB}^K$ in $\left[\underline{\eta}, \overline{\eta}\right)$ and $q_t^K \le q_{FB}^K$ at $\overline{\eta}$ is equivalent to just proving $q_t^K \le q_{FB}^K$ at $\overline{\eta}$. Because $\epsilon_t^B = 0$ at $\overline{\eta}$ (boundary condition), $R_t - \delta = \rho$ at $\eta_t = \overline{\eta}$ (see (20)). Therefore, from (11), at $\eta_t = \overline{\eta}$, the value of capital is given by

$$q_t^K = \frac{\alpha - \rho l_t + \lambda\left[q_t^K F(l_t) - l_t\right]}{\rho + \delta - \left(\mu_t^K + \sigma\sigma_t^K\right)}\,. \tag{A.25}$$

From (12),

$$q_{FB}^K = \frac{\alpha - \rho l_{FB} + \lambda\left[q_{FB}^K F(l_{FB}) - l_{FB}\right]}{\rho + \delta}\,. \tag{A.26}$$

By Itô's lemma,

$$\mu_t^K = \frac{dq_t^K}{d\eta_t}\frac{1}{q_t^K}\left(\mu_t^\eta \eta_t\right) + \frac{1}{2}\frac{d^2 q_t^K}{d\eta_t^2}\frac{1}{q_t^K}\left(\sigma_t^\eta \eta_t\right)^2 \tag{A.27}$$

Next, I prove that $\mu_t^K \le 0$ at $\eta_t = \overline{\eta}$. The first term is zero because $\frac{dq_t^K}{d\eta_t} = 0$ at $\eta_t = \overline{\eta}$ (boundary condition). I show that the second term is non-positive (i.e., $\frac{d^2 q_t^K}{d\eta_t^2} \le 0$ at $\eta_t = \overline{\eta}$). There exists an increasing sequence of $\{\eta_n\}_{n=1}^{+\infty}$ such that $\lim_{n\to\infty} \eta_n = \overline{\eta}$ and, $\forall n$, $\frac{d^2 q_t^K}{d\eta_t^2} < 0$ at η_n. This sequence can be constructed recursively. Given a small number ϵ_1, there exists at least one $\eta_1 \in [\overline{\eta} - \epsilon_1, \overline{\eta})$ such that $\frac{d^2 q_t^K}{d\eta_t^2} < 0$ at η_1, because $\frac{dq_t^K}{d\eta_t} > 0$ in $[\overline{\eta} - \epsilon_1, \overline{\eta})$ (Proposition 6) and $\frac{dq_t^K}{d\eta_t} = 0$ at $\overline{\eta}$. Next, for any integer $n > 1$, given ϵ_n such that $\overline{\eta} - \epsilon_n > \eta_{n-1}$, there exists at least one $\eta_n \in [\overline{\eta} - \epsilon_n, \overline{\eta})$ such that $\frac{d^2 q_t^K}{d\eta_t^2} < 0$ at η_n, again because $\frac{dq_t^K}{d\eta_t} > 0$ in $[\overline{\eta} - \epsilon_n, \overline{\eta})$ (Proposition 6) and $\frac{dq_t^K}{d\eta_t} = 0$ at $\overline{\eta}$. Note that, by construction, $\eta_n \ge \overline{\eta} - \epsilon_n > \eta_{n-1}$. As long as $\frac{d^2 q_t^K}{d\eta_t^2}$ is a continuous function of η_t, $\lim_{n\to\infty} \eta_n = \overline{\eta}$ and, $\forall n$, $\frac{d^2 q_t^K}{d\eta_t^2} < 0$ at η_n together imply that $\frac{d^2 q_t^K}{d\eta_t^2} \le 0$ at the limit $\overline{\eta}$; otherwise, there exists a N (sufficiently large) such that, $\forall n > N$, $\frac{d^2 q_t^K}{d\eta_t^2} > 0$ at η_n (which is in a small neighbourhood of $\overline{\eta}$, contradicting the definition of $\{\eta_n\}_{n=1}^{+\infty}$.

By Itô's lemma,

$$\sigma_t^K = \frac{dq_t^K}{d\eta_t}\frac{1}{q_t^K}\left(\sigma_t^\eta \eta_t\right)\,, \tag{A.28}$$

which is zero at $\eta_t = \overline{\eta}$ because $\frac{dq_t^K}{d\eta_t} = 0$ (boundary condition). Given $\mu_t^K \le 0$ and $\sigma_t^K = 0$, the denominator of $q^K(\overline{\eta})$ is greater than or equal to the denominator of q^K_{FB}.

I apply Lemma A.1 this result to comparing two values of y, q^K_t and q^K_{FB}, that correspond, respectively, to two values of x, $\rho + \delta - \left(\mu^K_t + \sigma\sigma^K_t\right)$ and $\rho + \delta$, with

$$H(y) = \max_l -\rho l + \lambda \left[yF(l) - l\right] \ . \tag{A.29}$$

By the envelope theorem, I obtain

$$H'(y) = \lambda F(l) \ , \tag{A.30}$$

where l is the optimal solution. Using (2), i.e., Parameter Restriction I, and the previously proved $\mu^K_t \leq 0$ and $\sigma^K_t = 0$, I obtain

$$H'(y) < \rho + \delta \leq \rho + \delta - \left(\mu^K_t + \sigma\sigma^K_t\right) \ . \tag{A.31}$$

Therefore, from Lemma A.1, $\rho + \delta \leq \rho + \delta - \left(\mu^K_t + \sigma\sigma^K_t\right)$ implies $q^K_t \leq q^K_{FB}$ at $\overline{\eta}$. According to Proposition 6, $q^K_t \leq q^K_{FB}$ at $\overline{\eta}$ implies $q^K_t < q^K_{FB}$ in $\left[\underline{\eta}, \overline{\eta}\right)$.

Proof of Proposition 9. The investment function $F(\cdot)$ is concave. I will first consider the case of strictly concave $F(\cdot)$ and then the case of linear $F(\cdot)$. Under $F'' < 0$, the results in Proposition 8 that $q^K_t < q^K_{FB}$ in $\left[\underline{\eta}, \overline{\eta}\right)$ and, at $\overline{\eta}$, $q^K_t \leq q^K_{FB}$ implies that $l^*_t < l_{FB}$ (l^*_t defined in (22) and l_{FB} defined in (13)) in $\left[\underline{\eta}, \overline{\eta}\right)$ and, at $\overline{\eta}$, $l^*_t \leq l_{FB}$. In the main text, I show that $l_t < l^*$ in $\left[\underline{\eta}, \overline{\eta}\right)$ and, at $\overline{\eta}$, $l_t = l^*_t$ without the pledgeability constraint. In the presence of the pledgeability constraint, $l_t \leq l^*_t$ at $\overline{\eta}$. Next, I consider linear $F(l) = \kappa l$. Under $\lambda(q^K_{FB}\kappa - 1) > \rho$ (implied by the parameter condition $\lambda(\frac{\alpha\kappa}{\rho+\delta} - 1) > \rho$, $q^K_t \geq \frac{\alpha\kappa}{\rho+\delta}$ in Proposition A.1, and $q^K_{FB} > q^K_t$ in Proposition 8)), I obtain $l_{FB} = \bar{l}$. Since $\bar{l} > q^K_{FB}$ and $q^K_{FB} \geq q^K_t$ (Proposition 8), we know that when the pledgeability constraint binds, $l_t = q^K_t < \bar{l} = l_{FB}$. When the pledgeability constraint does not bind, $l_t < q^K_t \leq q^K_{FB} < \bar{l} = l_{FB}$.

Proof of Lemma 6. Please refer to the main text.

Proof of Proposition 10. In the region where the entrepreneurs' pledgeability constraint does not bind, x_t is given by (24) and, as discussed in the main text, x_t is increasing in η_t. I characterize the range of η_t where the pledgeability constraint binds and how bank leverage behaves in this region. Lemma 3 shows that $dq^B(\eta_t)/d\eta_t = 0$ at $\eta_t = \overline{\eta}$, the upper boundary where bankers are not financially constrained and decides to consume rather than retain equity. Therefore, when η_t increases to a sufficiently high level and approaches $\overline{\eta}$, ϵ_t^B monotonically declines zero, implying that x_t, given by (24), approaches infinity. Note that bank lending per unit of capital is $l_t = x_t E_t / K_t = x_t \eta_t$. Therefore, as η_t approaches $\overline{\eta}$ driving x_t to infinity, we have l_t going to infinity as well, which violates the pledgeability constraint $l_t \leq q_t^K$.[47] Therefore, the solution of x_t given by (24) when the pledgeability constraint does not bind cannot hold when η_t is sufficiently close to $\overline{\eta}$ and $-\epsilon^B(\eta_t)$ in the denominator is close to zero; in other words, the pledgeability constraint must bind when η_t is close to $\overline{\eta}$, so we have $l_t = q_t^K$ (i.e., $x_t = \frac{q_t^K}{\eta_t}$) near $\overline{\eta}$ and the pledgeability constraint binds. Finally I show x_t is decreasing in η_t near $\overline{\eta}$. From Proposition 6, $\frac{dq^K(\overline{\eta})}{d\eta_t} = 0$, so, when η_t is close to $\overline{\eta}$, the numerator of x_t in (25) is increasing at a slower rate than the denominator, so x_t is decreasing in η_t.

Proof of Proposition 11. When the pledgeability constraint binds, bank ROE is solved by substituting out $R_t - \delta - r_t$ with $\gamma_t^B \sigma$ (see (18), bankers' optimality condition) and, in turn, γ_t^B with $-\epsilon_t^B (x_t - 1)\sigma$ (see (19)):

$$ROE_t = r_t + (R_t - \delta - r_t)\, x_t = \rho + \sigma^2 \left(-\epsilon_t^B\right)(x_t - 1)\, x_t \tag{A.32}$$

From Lemma 7, ϵ_t^B increases in η_t. From Proposition 10, when η_t is sufficiently large, x_t decreases in η_t, which implies the quadratic form $(x_t - 1)\, x_t$ decreases in η_t because $x_t > 1$. Therefore, bank ROE decreases in η_t when entrepreneurs' pledgeability constraint binds and η_t is sufficiently large.

Proof of Lemma 8. Following Brunnermeier and Sannikov (2014), I derive the stationary probability density. Probability density of η_t at time t, $p(\eta, t)$, has Kolmogorov forward equation

$$\frac{\partial}{\partial t} p(\eta, t) = -\frac{\partial}{\partial \eta}\left(\eta \mu^\eta(\eta)\, p(\eta, t)\right) + \frac{1}{2}\frac{\partial^2}{\partial \eta^2}\left(\eta^2 \sigma^\eta(\eta)^2\, p(\eta, t)\right).$$

Note that in a Markov equilibrium, μ_t^η and σ_t^η are functions of η_t. A stationary density is a solution to the forward equation that does not vary with time (i.e. $\frac{\partial}{\partial t} p(\eta, t) = 0$). So I suppress the time variable, and denote stationary density as $p(\eta)$. Integrating the forward equation over η, $p(\eta)$

[47] Capital value, $q_t^K = q^K(\eta_t)$, is continuous and differentiable function of $\eta_t \in \left[\underline{\eta}, \overline{\eta}\right]$ according to Proposition 8.

solves the following first-order ordinary differential equation within the two reflecting boundaries:

$$0 = C - \eta \mu^{\eta}(\eta)\, p(\eta) + \frac{1}{2}\frac{d}{d\eta}\left(\eta^2 \sigma^{\eta}(\eta)^2 p(\eta)\right), \quad \eta \in \left[\underline{\eta}, \overline{\eta}\right].$$

The integration constant C is zero because of the reflecting boundaries. The boundary condition for the equation is the requirement that probability density is integrated to one (i.e. $\int_{\underline{\eta}}^{\overline{\eta}} p(\eta)\, d\eta = 1$).

Proof of Proposition 12. At $\overline{\eta}$, I use (18) and (19) to substitute $R(\overline{\eta}) - \delta - \rho$ with $-\epsilon^B(\overline{\eta})(x(\overline{\eta}) - 1)\sigma^2$ in $\mu^{\eta}(\eta)$ (given by (A.40)), and, form (A.3), $\epsilon^B(\overline{\eta}) = 0$. Therefore, I obtain

$$\mu^{\eta}(\overline{\eta}) = \rho - [\lambda F(l(\overline{\eta})) - \delta] - (x(\overline{\eta}) - 1)\sigma^2. \tag{A.33}$$

Substituting this expression of $\mu^{\eta}(\overline{\eta})$ and $\frac{d\sigma^{\eta}(\overline{\eta})}{d\eta} = x'(\overline{\eta})\sigma$ into (A.39), I obtain

$$\frac{p'(\overline{\eta})}{p(\overline{\eta})} = \frac{2\overline{\eta}}{\overline{\eta}^2 (x(\overline{\eta}) - 1)^2 \sigma^2}\left\{\rho - [\lambda F(x(\overline{\eta})\overline{\eta}) - \delta] - (x(\overline{\eta}) - 1)\sigma^2 [x(\overline{\eta}) + x'(\overline{\eta})\overline{\eta}]\right\}, \tag{A.34}$$

where $l(\overline{\eta})$ is substituted with $x(\overline{\eta})\eta$ (i.e., the loan market clearing condition) and $\sigma^{\eta}(\overline{\eta})$ is substituted with $(x(\overline{\eta}) - 1)\sigma$ (see (A.41)). Note that the loan market clearing condition, $x(\eta)\eta = l(\eta)$, implies that $x(\overline{\eta}) + x'(\overline{\eta})\overline{\eta} = l'(\eta)$. Therefore, equation (A.34) can be simplified to

$$\frac{p'(\overline{\eta})}{p(\overline{\eta})} = \frac{2\overline{\eta}}{\overline{\eta}^2 (x(\overline{\eta}) - 1)^2 \sigma^2}\left\{\rho - [\lambda F(x(\overline{\eta})\overline{\eta}) - \delta] - (x(\overline{\eta}) - 1)\sigma^2 l'(\overline{\eta})\right\}. \tag{A.35}$$

Because the pledgeability constraint binds near $\overline{\eta}$, i.e., $l(\eta) = q^K(\eta)$ and $\frac{dq^K(\eta)}{d\eta} = 0$ at $\overline{\eta}$ (see Proposition 6), $l'(\overline{\eta})$ is equal to zero. Therefore, equation (A.35) can be simplified to

$$\frac{p'(\overline{\eta})}{p(\overline{\eta})} = \frac{2\overline{\eta}}{\overline{\eta}^2 (x(\overline{\eta}) - 1)^2 \sigma^2}\left\{\rho - [\lambda F(x(\overline{\eta})\overline{\eta}) - \delta]\right\} \geq \frac{\overline{\eta}}{\overline{\eta}^2 (x(\overline{\eta}) - 1)^2} > 0, \tag{A.36}$$

where the second inequality follow the parameter condition given by (2). Therefore, given the continuity of $p'(\eta)$, there exists a neighborhood of $\overline{\eta}$, denoted by $(\eta_B, \overline{\eta}]$, where $p'(\eta) > 0$, and thus, the maximum value of $p(\eta)$ in this neighborhood is obtained at $\overline{\eta}$.

Proof of Lemma 9. The expected time to reach from $\underline{\eta}$ is solved below. Define $f_{\eta_0}(\eta)$ the expected time it takes to reach η_0 starting from $\eta \leq \eta_0$. Define $g(\eta_0) = f_{\eta_0}(\underline{\eta})$ the expected time to reach η_0 from $\underline{\eta}$. One has to reach $\eta \in (\underline{\eta}, \eta_0)$ first and then reach η_0 from η. Therefore, $g(\eta) + f_{\eta_0}(\eta) = g(\eta_0)$. Since $g(\eta_0)$ is constant, we differentiate both sides to have $g'(\eta) = -f'_{\eta_0}(\eta)$ and $g''(\eta) = -f''_{\eta_0}(\eta)$. From η_t, the expected time to reach η_0, denoted by $f_{\eta_0}(\eta_t)$, is decomposed into $s-t$, and $E_t[f_{\eta_0}(\eta_s)]$, i.e., the expected time to reach η_0 from η_s $(s \geq t)$ after $s-t$ has passed. We have $f_{\eta_0}(\eta_t)$ equal to $E_t[f_{\eta_0}(\eta_s)] + s - t$. Therefore, $t + f_{\eta_0}(\eta_t)$ is a martingale, so f_{η_0} satisfies the ordinary differential equation: $1 + f'_{\eta_0}(\eta)\mu^\eta(\eta) + \frac{\sigma^\eta(\eta)^2}{2} f''_{\eta_0}(\eta) = 0$. Therefore, $g(\eta)$ must satisfy

$$1 - g'(\eta)\mu^\eta(\eta) - \frac{\sigma^\eta(\eta)^2}{2} g''(\eta) = 0.$$

It takes no time to reach $\underline{\eta}$, so $g(\underline{\eta}) = 0$. Moreover, since $\underline{\eta}$ is a reflecting boundary, $g'(\underline{\eta}) = 0$.

Proof of Proposition 13. Consider z in the neighborhood of $\underline{\eta}$, $(\underline{\eta},\ \underline{\eta} + \epsilon)$, where ϵ is a positive constant. The boundary condition $g'(\underline{\eta}) = 0$ implies that, first, $g''(\underline{\eta}) = \frac{2}{(x(\underline{\eta})-1)\sigma^2}$ (from the ODE in Lemma 9) where I use the expression of diffusion of η_t in (16), i.e., $\sigma^\eta(\eta) = (x(\eta) - 1)\sigma$ $x(\underline{\eta})$ at $\eta_t = \underline{\eta}$, and, second, by Taylor's expansion and the boundary condition $g'(\underline{\eta}) = 0$,

$$g'(z) = g'(\underline{\eta}) + g''(\underline{\eta})(z - \underline{\eta}) + o(z - \underline{\eta}) = \frac{2(z - \underline{\eta})}{(x(\underline{\eta}) - 1)\sigma^2} + o(z - \underline{\eta}) \ , \tag{A.37}$$

where let $o(z - \underline{\eta})$ denote the higher order infinitesimal of $z - \underline{\eta}$, which is smaller than the width of the neighborhood ϵ. Therefore, for any $\eta \in (\underline{\eta},\ \underline{\eta} + \epsilon)$, I obtain the expected time it takes to reach η from $\underline{\eta}$:

$$g(\eta) = g(\underline{\eta}) + \int_{\underline{\eta}}^{\eta} g'(z)\,dz = \frac{(\eta - \underline{\eta})^2}{(x(\underline{\eta}) - 1)\sigma^2} + o\left((\eta - \underline{\eta})^2\right) \ , \tag{A.38}$$

where I use the boundary condition, $g(\underline{\eta}) = 0$, in Lemma 9. Equation (A.38) shows that in a small neighborhood of $\underline{\eta}$, the recovery time increases quadratically in the destination state η (in the numerator). More importantly, the recovery time is longer when bank leverage, $x(\underline{\eta})$, is lower (in the denominator). Let η_R denote $\underline{\eta} + \epsilon$.

Proof of Proposition 14. From the ODE (27) that solves $p(\eta)$, I obtain

$$\frac{p'(\eta)}{p(\eta)} = \frac{2}{(\sigma^\eta(\eta)\,\eta)^2}\left[\mu^\eta(\eta)\,\eta - \sigma^\eta(\eta)^2\,\eta - \sigma^\eta(\eta)\,\eta^2\frac{d\sigma^\eta(\eta)}{d\eta}\right]. \tag{A.39}$$

From (16) and under the equilibrium condition $r_t = \rho$, $\mu^\eta(\eta)$ is given by

$$\mu^\eta(\eta) = \rho + x(\eta)\left(R(\eta) - \delta - \rho\right) - \left[\lambda F\left(l(\eta)\right) - \delta\right] - \left(x(\eta) - 1\right)\sigma^2, \tag{A.40}$$

and $\sigma^\eta(\eta)$ is given by

$$\sigma^\eta(\eta) = \left(x(\eta) - 1\right)\sigma, \tag{A.41}$$

which implies that $\frac{d\sigma^\eta(\eta)}{d\eta} = x'(\eta)\,\sigma$. At $\underline{\eta}$ where the pledgeability constraint does not bind, I use (18) and (19) to substitute $R\left(\underline{\eta}\right) - \delta - \rho$ with $-\epsilon^B\left(\underline{\eta}\right)\left(x\left(\underline{\eta}\right) - 1\right)\sigma^2$ in $\mu^\eta(\eta)$, and, form (A.2), $\epsilon^B\left(\underline{\eta}\right) = -1$. Therefore, I obtain

$$\mu^\eta\left(\underline{\eta}\right) = \rho - \left[\lambda F\left(l\left(\underline{\eta}\right)\right) - \delta\right] + \left(x\left(\underline{\eta}\right) - 1\right)^2\sigma^2 = \rho - \left[\lambda F\left(l\left(\underline{\eta}\right)\right) - \delta\right] + \sigma^\eta\left(\underline{\eta}\right)^2. \tag{A.42}$$

Substituting this expression of $\mu^\eta\left(\underline{\eta}\right)$ and $\frac{d\sigma^\eta\left(\underline{\eta}\right)}{d\eta} = x'\left(\underline{\eta}\right)\sigma$ into (A.39), I obtain

$$\frac{p'\left(\underline{\eta}\right)}{p\left(\underline{\eta}\right)} = \frac{2\underline{\eta}}{\underline{\eta}^2\left(x\left(\underline{\eta}\right) - 1\right)^2\sigma^2}\left\{\rho - \left[\lambda F\left(x\left(\underline{\eta}\right)\underline{\eta}\right) - \delta\right] - \underline{\eta}\sigma^\eta\left(\underline{\eta}\right)\sigma x'\left(\underline{\eta}\right)\right\}, \tag{A.43}$$

where $l\left(\underline{\eta}\right)$ is substituted with $x\left(\underline{\eta}\right)\eta$ (i.e., the loan market clearing condition). Next, I further simplify the expression by solving $x'\left(\underline{\eta}\right)$.

From (24), $\epsilon^B\left(\underline{\eta}\right) = -1$ and $\frac{dq^K\left(\underline{\eta}\right)}{d\eta} = 0$, I obtain

$$x'\left(\underline{\eta}\right) = \left(\frac{\lambda\left(q^K\left(\underline{\eta}\right)\kappa - \rho\right) - \rho}{\sigma^2}\right)\frac{d\epsilon^B\left(\underline{\eta}\right)}{d\eta}. \tag{A.44}$$

From (24) and (A.41), I obtain

$$x'\left(\underline{\eta}\right) = \left(x\left(\underline{\eta}\right) - 1\right)\frac{d\epsilon^B\left(\underline{\eta}\right)}{d\eta}. \tag{A.45}$$

Substituting this expression into (A.43), I obtain

$$\begin{aligned}\frac{p'\left(\underline{\eta}\right)}{p\left(\underline{\eta}\right)} =& \frac{2\underline{\eta}}{\underline{\eta}^2\left(x\left(\underline{\eta}\right)-1\right)^2\sigma^2}\left\{\rho-\left[\lambda F\left(x\left(\underline{\eta}\right)\underline{\eta}\right)-\delta\right]-\underline{\eta}\sigma^\eta\left(\underline{\eta}\right)^2\frac{d\epsilon^B\left(\underline{\eta}\right)}{d\eta}\right\} \\ \le& \frac{2\underline{\eta}}{\underline{\eta}^2\left(x\left(\underline{\eta}\right)-1\right)^2\sigma^2}\left\{\rho-\left[\lambda F\left(x\left(\underline{\eta}\right)\underline{\eta}\right)-\delta\right]-2\tau+\sigma^2+\sigma^\eta\left(\underline{\eta}\right)^2\right\} \\ \le& \frac{2\underline{\eta}}{\underline{\eta}^2\left(x\left(\underline{\eta}\right)-1\right)^2\sigma^2}\left(\rho+\delta-2\tau+\sigma^2+\sigma^\eta\left(\underline{\eta}\right)^2\right) , \end{aligned} \tag{A.46}$$

where the first inequality follows (A.12) and the second follows $l\left(\underline{\eta}\right) \geq 0$ (and $F'(\cdot) > 0$).

Finally, we examine the upper bound of $\frac{p'\left(\underline{\eta}\right)}{p\left(\underline{\eta}\right)}$ given by (A.46). Equation (24) and $\epsilon^B\left(\underline{\eta}\right) = -1$ imply that

$$\sigma^\eta\left(\underline{\eta}\right)^2 = \frac{\left(\lambda\left[q^K\left(\underline{\eta}\right)\kappa-1\right]-\rho\right)^2}{\sigma^2} . \tag{A.47}$$

Therefore, from (A.46), a sufficient condition for $p'\left(\underline{\eta}\right) < 0$ is

$$\rho+\delta-2\tau+\sigma^2+\frac{\left(\lambda\left[q^K\left(\underline{\eta}\right)\kappa-1\right]-\rho\right)^2}{\sigma^2} < 0 , \tag{A.48}$$

which is equivalent to

$$q^K\left(\underline{\eta}\right) < \frac{1}{\kappa}+\frac{\rho+\sigma\sqrt{2\tau-\rho-\delta-\sigma^2}}{\lambda\kappa} , \tag{A.49}$$

and it holds because $\lambda\left[q^K\left(\underline{\eta}\right)\kappa-1\right]-\rho \geq 0$ (see (23)). It has been proven that $p'\left(\underline{\eta}\right)$ is strictly negative. This implies that there exists a neighborhood of $\underline{\eta}$, denoted by $(\underline{\eta}, \eta_C)$ where $p'\left(\eta\right) < 0$. In this neighborhood, in the interval $\left[\underline{\eta},\ \eta_C\right)$, the stationary density, $p(\eta)$, is maximized at $\underline{\eta}$.

Proof of Proposition 15. At $\eta_t = \underline{\eta}$, $\epsilon_t^B = -1$ (see equation (A.2)), so $x_t = 1 + (R_t - \delta - \rho)/\sigma^2$ (see equation (8) and (24)). Therefore, when credit intervention strictly reduces the loan rate, R_t, it also strictly decreases bank leverage, x_t, at $\underline{\eta}$. Since bank leverage, x_t, is a continuous function of η_t, there exists a neighborhood of $\underline{\eta}$ where credit intervention reduces bank leverage, x_t.

Proposition A.1 (Capital Value Lower Bound) $q_t^K \geq \frac{\alpha}{\rho+\delta}$.

Proof of Proposition A.1. From Proposition 6, $q_t^K \geq q^K\left(\underline{\eta}\right)$. Therefore, it is sufficient to prove $q^K\left(\underline{\eta}\right) \geq \frac{\alpha}{\rho+\delta}$. At $\underline{\eta}$, by Itô's lemma, we have

$$\mu^K\left(\underline{\eta}\right) = \frac{dq^K\left(\underline{\eta}\right)}{d\eta}\frac{\mu^\eta\left(\underline{\eta}\right)\underline{\eta}}{q^K\left(\underline{\eta}\right)} + \frac{d^2q^K\left(\underline{\eta}\right)}{d\eta^2}\frac{\left(\sigma^\eta\left(\underline{\eta}\right)\underline{\eta}\right)^2}{2q^K\left(\underline{\eta}\right)} = \frac{d^2q^K\left(\underline{\eta}\right)}{d\eta^2}\frac{\left(\sigma^\eta\left(\underline{\eta}\right)\underline{\eta}\right)^2}{2q^K\left(\underline{\eta}\right)}\,, \tag{A.50}$$

where the second equation follows the boundary condition that $\frac{dq^K\left(\underline{\eta}\right)}{d\eta} = 0$ in Proposition 6. Because, according to Proposition 6, $\frac{dq^K(\eta)}{d\eta} > 0$ when $\eta > \underline{\eta}$, I obtain $\frac{d^2q^K\left(\underline{\eta}\right)}{d\eta^2} > 0$, and thus, $\mu^K\left(\underline{\eta}\right) > 0$. Using the capital valuation equation (11) in Lemma 2, I obtain

$$\begin{aligned} q^K\left(\underline{\eta}\right) &= \frac{\alpha - \left(R\left(\underline{\eta}\right)-\delta\right)l\left(\underline{\eta}\right) + \lambda\left[q^K\left(\underline{\eta}\right)F\left(l\left(\underline{\eta}\right)\right) - l\left(\underline{\eta}\right)\right]}{\rho - \left(\mu^K\left(\underline{\eta}\right) - \delta + \sigma\sigma^K\left(\underline{\eta}\right)\right)} \\ &= \frac{\alpha - \left(R\left(\underline{\eta}\right)-\delta\right)l\left(\underline{\eta}\right) + \lambda\left[q^K\left(\underline{\eta}\right)F\left(l\left(\underline{\eta}\right)\right) - l\left(\underline{\eta}\right)\right]}{\rho - \left(\mu^K\left(\underline{\eta}\right) - \delta\right)} \geq \frac{\alpha}{\rho - \left(\mu^K\left(\underline{\eta}\right) - \delta\right)} \geq \frac{\alpha}{\rho+\delta}\,, \end{aligned} \tag{A.51}$$

where the second equality follows $\frac{dq^K\left(\underline{\eta}\right)}{d\eta} = 0$ in Proposition 6 and, by Itô's lemma,

$$\sigma^K\left(\underline{\eta}\right) = \frac{dq^K\left(\underline{\eta}\right)}{d\eta}\frac{\sigma^\eta\left(\underline{\eta}\right)\underline{\eta}}{q^K\left(\underline{\eta}\right)} = 0\,, \tag{A.52}$$

the first inequality follows the fact that the numerator is greater under the optimal $l\left(\underline{\eta}\right)$ than under $l = 0$, and finally, the second inequality follows from $\mu^K\left(\underline{\eta}\right) > 0$, which was previously proven.

B Equity Issuance Costs

Bank equity issuance cost plays an important role in the model. Without this friction, the credit supply curve becomes perfectly elastic at interest rate $\rho + \delta$, where ρ (households' discount rate) accounts for the time value of money and δ is the expected default probability. Under the equity issuance cost, banks become endogenously risk-averse and charges a loan risk premium. The elasticity of credit supply increases in the level of bank equity. The cost of equity issuance, χ, is set to 10% in the numerical solution based on evidence that I review below.

Seasoned equity offering (SEO) incurs three types of costs: (1) the direct costs, including underwriters' compensation and other direct costs (e.g., registration, legal, and auditing fees); (2) underpricing; (3) the announcement effect on total market capitalization (Ritter, 2003). The underwriters receive compensation for arranging and underwriting an offering of equity securities. The underwriters' compensation takes the form of "gross spread", which is the difference between the underwriting price received by the issuing company and the actual price offered to the public. Boyson, Fahlenbrach, and Stulz (2016) calculate that the average gross spread is 5.02% of issuance proceeds for banks. This number is close to the average of 5.44% across different industries (Lee, Lochhead, Ritter, and Zhao, 1996). The other direct costs are estimated to be 1.7% (Lee, Lochhead, Ritter, and Zhao, 1996). In sum, the total direct costs is close to 7% of the issuance proceeds.

The second type of issuance costs is the indirect cost of underpricing, which is estimated to be 2.2% of the issuance proceeds averaged across industries (Corwin, 2003). Underpricing is the practice of listing an equity offering at a price below its marketable value. Smith (1977) was the first to document significant underpricing for SEOs. Corwin (2003) provides a review of the theoretical literature of underpricing that involve a variety of model ingredients, such as uncertainty and asymmetric information, price pressure, manipulative trading, transaction costs, and conventional derwriting practices such as rounding offer prices.

The sum of direct costs and underpricing is around to 9% of the issuance proceeds. The parameter of proportional issuance cost, χ, is set to 10% in the numerical solution, which is 1% above the sum of direct costs and underpricing. This wedge is intended to capture the announcement effect in a conservative fashion. According to the signaling theory in Myers and Majluf (1984), equity issuance triggers a negative response in the stock market as it signals firm overvaluation. A large literature on the SEO announcement effect supports the theory. The estimate of announcement effect is around 2% across industries in the U.S. (Ritter, 2003). Because the announcement effect is on the issuing firm's total market capitalization, it can be very large relative to the issuance proceeds. For example, given an announcement effect of 2%, an equity offering that is 3.5% of total market capitalization (the sample average from Baron (2020)) triggers a negative market response that amounts to 57% ($= 2\%/3.5\%$) of the issuance proceeds.

However, whether the announcement effect should be regarded as issuance cost is a subject of debate. One can argue that the negative market response would have occurred without equity issuance because the management's opinion regarding firm value may very well be disclosed in some other manner. Therefore, in the calibration of issuance cost, χ, I only set a very value of 1% of issuance proceeds to capture the announcement effect. Related, for banks, equity issuance can be motivated by regulatory compliance, so the signaling model may not directly applicable. Keeley (1989), for example, argues that bank regulation reduces the information content that otherwise would be revealed by a security issuance (in general negative), and consequently stock announcement effects might be less negative for bank SEOs than those of non-banks.[48] Li, Liu, Siganos, and Zhou (2016) find the announcement effect of bank equity issuance is around -1%. The passage of Dodd-Frank Act strengthened regulatory requirement on equity capital and thus diminished the negative signaling effect of banks' decision to replenish capital through equity issuance.

C Parameter Calibration

The numerical solutions are based on the following parameter values. One unit of time is one year. Goods produced per unit of capital, α, is normalized to one. Under the constant return-to-scale production technology, capital represents efficiency units of production. Following Brunnermeier and Sannikov (2014), I adopt the following parameter values for capital depreciation rate, the size of shock to capital, and households' discount rate, setting δ to 0.05, σ to 0.1, and ρ to 0.05, respectively. I adopt these commonly used parameter values in the literature so that I can attribute the novel dynamics in my model to its internal mechanisms rather than drastically different parameter values. The equity issuance cost, $\chi = 0.1$, is based on the empirical literature (Appendix B).

Bank discount rate, $\overline{\rho}$, which is equal to the expected growth rate of value function in equilibrium, is set to 0.1 in line with the average stock return of US banks.[49] Note that the Hamilton-Jacobi-Bellman (HJB) equation (A.1) in Appendix A shows that the expected growth rate of bank value function, i.e., bank shareholders' value, is equal to $\overline{\rho}$.[50]

The parameters, λ and κ are related to firms' credit needs and how profitable it is to create new productive capital. For the lack of direct counterparts in the literature, I calibrate these parameters targeting leverage (debt-to-total assets ratio) and valuation (equity price-to-earnings ratio) of

[48] Consistent with this hypothesis, the announcement effect is smaller for banks than non-bank firms. The earlier studies documented a market negative response of 1.4% to 1.5% (Wall and Peterson, 1988; Keeley, 1989; Wansley and Dhillon, 1989). The Federal Deposit Insurance Corporation Improvement Act (FDICIA) was enacted in 1991. Krishnan et al. (2010) find that the announcement effect diminished after FDICIA and regulation-driven equity raising is associated with a smaller negative announcement effect. The difference in announcement effects between voluntary and involuntary equity issuance is also found in earlier studies Cornett and Tehranian (1994).

[49] The Dow Jones U.S. Banks Index is designed to measure the stock performance of U.S. banks (Bloomberg Ticker: DJUSBK). As of October 5th, 2025, the ten-year average annualized return is 9.22%.

[50] The calibration of $\overline{\rho}$ is also consistent with Gertler and Kiyotaki (2010) who set a bank exit rate to 0.1, as idiosyncratic Poisson-arriving exit is isomorphic to exponential discounting.

US nonfinancial firms that reflect, respectively, firms' demand for debt financing and profits from creating efficiency units of production (capital). Specifically, I set λ to 0.01 and κ to 0.65 such that the average firm leverage and equity price-to-earnings match data.[51] Li (2025) reports a ratio of bank loans to total assets equal to 19% (see also Lian and Ma (2020)).[52] Ben-David and Chinco (2024) reports a price-to-earnings ratio of 16 for the median firm. For the decreasing return-to-scale investment technology, parameters are set so that it stays close to the baseline investment technology with constant return-to-scale, so I set κ and κ_1 to 0.66 and 0.99, respectively.

The parameters discussed so far are for the numerical solutions of the laissez-faire economy in Figure 4. The numerical solution of the economy under credit intervention in Figure 5 requires the following parameters of credit policy, η_G, ω_0, and ω_1. The threshold, η_G, is set to 0.5, so the government intervenes at around -1% growth rate of the economy. For comparison, the annual growth rate of US real GDP in 2020 was -1.02%, when credit intervention in the U.S. took place in response to the Covid-19 pandemic. As a reminder, the government intervenes when the private-sector credit intermediation capacity, given by η_t, is below η_G. There is no direct empirical counterpart for the threshold, so η_G is set to a reasonable value for the purpose of demonstrating the qualitative predictions of the model rather than to perfectly mimic policy in practice. Moreover, I set ω_0 and ω_1 to 0.97 and 1.95, respectively. The first discipline for these two parameters is to have the scale of intervention equal to zero at the threshold η_G. The second discipline is from the observed scale of intervention during the Covid-19 pandemic in the U.S. Under these parameters, the scale of credit intervention is 11% of firms' total debt when the growth rate of the economy is -3%. For comparison, the (annualized) quarterly growth rate of US real GDP in 2020 ranged from -7.5% to 1.3%. The scale of intervention in 2020 is 1.35 trillion dollars, which is 11% of the total debt of U.S. nonfinancial corporate business that is around 12 trillion dollars (FRED, 2024).[53]

[51] The average values are calculated under the stationary distribution of η_t that will be formally introduced later. The debt-to-asset ratio is l_t/q_t^K, i.e., the ratio of credit per unit of capital scaled by capital value. The equity price-to-earnings ratio is computed as follows. One unit of capital is worth q_t^K with debt l_t, so equity is $q_t^K - l_t$. Annual expected ("forward") earnings from one unit of capital is given by $\alpha - \delta q_t^K - (R_t - \delta)l_t$, where the first term is the production flow, the second term represents the expected capital destruction, and the last term represents debt costs.

[52] See Table D.1 in Li (2025) where firms are classified into those that have tangible (collateral) assets and rely on bank credit to finance investment and those whose production is more dependent on intangibles and investment financed by internal cash holdings. The former category has a bank loan-to-total assets ratio of 0.19.

[53] The Fed set up around 2.3 trillion credit support decomposed into 0.6 trillion in MSLP, 0.75 trillion in PMCCF and SMCCF, and 0.95 trillion in PPP Liquidity Facility (Li and Li, 2025). I sum up the credit support via MSLP, PMCCF, and SMCCF and leave out PPP which is essentially subsidy rather than government lending to firms at market rates.

D Welfare

The welfare at time t is given by (29). Conjecture that $W_t = \theta(\eta_t)K_t$, where K_t evolves as

$$\frac{dK_t}{K_t} = \mu_t^K dt + \sigma dZ_t, \tag{D.53}$$

where $\mu_t^K = F(l_t)\lambda - \delta$. I conjecture that in equilibrium, $\theta_t = \theta(\eta_t)K$ evolves as

$$d\theta_t = \mu_t^\theta \theta_t dt + \sigma_t^\theta \theta_t dZ_t, \tag{D.54}$$

where μ_t^θ and σ_t^θ will be determined endogenously. Given the law of motion of K_t and θ_t, the following HJB equation holds:

$$\rho = (\alpha - \lambda l_t)/\theta_t + \mu_t^\theta + \mu_t^K + \sigma_t^\theta \sigma = (\alpha - \lambda l_t)/\theta_t + \mu_t^\theta + F(l_t)\lambda - \delta + \sigma_t^\theta \sigma. \tag{D.55}$$

Rearranging the HJB equation, I obtain

$$\rho - (\mu_t^\theta - \delta + \sigma_t^\theta \sigma) = (\alpha - \lambda l_t)/\theta_t + F(l_t)\lambda, \tag{D.56}$$

where the right side contains θ_t and l_t that can be solved by the credit-market clearing condition as functions of η_t, the first derivative of $q^B(\eta_t)$, and $q^K(\eta_t)$(see Appendix C). Therefore, given the solutions of $q^B(\eta_t)$ and $q^K(\eta_t)$, the right side contains $\theta(\eta_t)$ and the rest is a function of η_t.

On the left side of the equation (D.56), by Itô's lemma, $\mu_t^\theta = \mu^\theta(\eta_t)$ can be written as

$$\mu^\theta(\eta_t) = \frac{d\theta(\eta_t)}{d\eta_t}\frac{\mu^\eta(\eta_t)\eta_t}{\theta(\eta_t)} + \frac{d^2\theta(\eta_t)}{d\eta_t^2}\frac{(\sigma^\eta(\eta_t)\eta_t)^2}{2\theta(\eta_t)}, \tag{D.57}$$

and $\sigma_t^\theta = \sigma^\theta(\eta_t)$ can be written as $\sigma^\theta(\eta_t) = \frac{d\theta(\eta_t)}{d\eta_t}\frac{\sigma^\eta(\eta_t)\eta_t}{\theta(\eta_t)}$, where $\mu^\eta(\eta_t)$ and $\sigma^\eta(\eta_t)$ are functions of η_t and given in the law of motion of η_t (16). Therefore, I have shown that the equation (D.56) is a second-order differential equation for $\theta(\eta_t)$. In summary, I conjecture $W_t = \theta(\eta_t)K_t$, and under this conjecture, I have shown that a function $\theta(\eta_t)$ exists as a solution to the second-order differential equation (D.56), thus confirming the conjecture. Solving the welfare function (a conditional expectation) is equivalently to solving the differential equation (D.56). This is similar to the classic result of Feynman–Kac formula.